Construction of QCD Hamiltonian on the Light Front, gauge invariantly regularized, with zero modes modeling the vacuum

M. Yu. Malyshev, <u>E. V. Prokhvatilov</u>

October 21, 2010

1 Introduction

The formulation of quantum field theory (QFT) on the light front (LF) uses the coordinates (P. A. M. Dirac. Forms of relativistic dynamics. Rev. Mod. Phys. 1949.)

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}, \quad x^{\perp} = (x^1, x^2) = x^k,$$

where x^0, x^1, x^2, x^3 are Lorentz coordinates, the x^+ plays the role of time, and the LF is defined by the eq-n $x^+ = 0$. This formulation leads formally to a possibility of simple description of vacuum state as the state with minimal eigenvalue $p_- = 0$ of the momentum operator P_- (the generator of translations in x^-):

$$P_- = \frac{P_0 - P_3}{\sqrt{2}}, P_- \geqslant 0 \quad for \quad p_0 \geqslant 0, \ p^2 \geqslant 0.$$

In usual, equal Lorentz time quantization, the complexity of the vacuum state description was the main difficulty in attempts to solve nonperurbatively the Schroedinger eq-n for QFT.

This advantage of the LF quantization allows to develop nonperturbative approach to calculations of mass spectrum of bound states (with a hope to apply this in QCD) solving the eigenvalue problem for the LF Hamiltonian $P_+ = \frac{P_0 + P_3}{\sqrt{2}}$ in the Fock space on the LF (A. M. Annenkova, V. A. Franke, E. V. Prokhvatilov. The solution of Schroedinger equation on the light front for Sine-Gordon model. Vestnik. Leningr. State. University. 1985. S. J. Brodsky, H.-C. V. Pauli, S. S. Pinsky. Quantum Chromodynamics and other Field Theories on the Light Cone. Phys. Reports. 1998.)

The canonical formulation of field theory on the LF is confronted with difficulties of treating zero modes of fields in x^-. For example, the kinetic term in the Lagrangian $L(x)$ for scalar field $\phi(x)$ has the following form in LF coordinates:

$$L = \partial_+\phi\,\partial_-\phi + \cdots\,,$$

where we write out the term with the derivative in x^+. Zero mode ($p_- = 0$, or the field independent of x^-) drops out of this term. So the canonically conjugated momentum is zero, and this mode is not independent dynamical variable, as distinct from other modes. Along with this peculiarity one gets singularities in the Hamiltonian at $p_- = 0$.

Two types of the regularization are usually applied:

(1) $|p_-| \geqslant \varepsilon > 0$,

(2) the so called "DLCQ" (Descretized Light Cone Quantization) regularization (S. J. Brodsky, H.-C. V. Pauli, S. S. Pinsky.), i.e. the cutoff in x^-, $|x^-| \leqslant L$, plus periodic boundary conditions on fields.

However the (1) is Lorentz and gauge nonsymmetric, and excludes zero modes in x^- which are important for correct description of vacuum effects.

The DLCQ also breakes Lorentz symmetry but supports gauge invariance. The p_- becomes descrete due to periodic boundary conditions. Well separated zero modes can be expressed in terms of other modes by solving corresponding canonical constraints. However for most models these constraints are too complicated, and the problem is practically unsolvable (V. A. Franke, Yu. V. Novozhilov, E. V. Prokhvatilov. On The Light Cone Formulation of Nonabelian Gauge Theory. Lett. Math. Phys. 1981) .

The violation of Lorentz symmetry in these regularizations can lead to problems with the renormalization of the theory and also to its possible nonequivalence with usual formulation in Lorentz coordinates. In the framework of perturbation theory it was shown that for restoring the equivalence it is necessary to introduce into the regularized Hamiltonian on the LF the additional terms like "counterterms" in the renormalization procedure (S. A. Paston, V. A. Franke. Comparison of quantum field perturbation theory for the light front with the theory in Lorentz coordinates. Theor. Math. Phys. 1997, hep-th/9901110.) .

To investigate the problem with the description of vacuum condensates on the LF we used the well known gauge field model in two dimensional space-time, the QED(1+1) . This model can be transformed into the scalar field model with nonpolynomial interaction, which can be treated on the LF better than the original gauge theory. As a result we have found the terms which one must add to canonical expression for the LF QED(1+1) Hamiltonian in order to make LF formulation equivalent to one in equal Lorentz time quantization. These terms contain all information about condensates and depend only on zero modes.

(S. A. Paston, E. V. Prokhvatilov, V. A. Franke. Theor. Math. Phys. 2002. hep-th/0302016. On the construction of corrected light-front Hamiltonian for QED_2. hep-th/0011224.)

Futhermore it was noticed that the same vacuum effects on the mass spectrum in QED(1+1) can be approximately obtained via special limiting transition to the LF Hamiltonian from the theories on the space-like hyperplanes close to the LF. One must go to the LF at fixed parameter L, which confines the x^- by the inequality $|x^-| \leqslant L$, and "freeze" the limiting transition for zero modes while the L remains finite. (E. V. Prokhvatilov, V. A. Franke. Approximate description of QCD condensates in light cone coordinates. Sov. J. Nucl. Phys. 1988).

We would like to apply analogous approximate way in Quantum Chromodynamics (QCD) as some semiphenomenological way to describe vacuum effects. With this aim we introduce gauge invariant regularization of QCD adapted for different treating of zero and nonzero modes. We use the lattice in "transverse" coordinates x^1, x^2 and use the unitary matrices as link variables to describe gluon zero modes. All other fields are related to lattice sites. Furthermore we introduce a cutoff in p_-. It is possible to give gauge invariant form to this description as well as for the cutoff in p_-.

Until the regularization is removed Lorentz symmetry is violated by the form of the regularization. The vacuum state, defined as an eigen state of the P_- with minimal value $p_- = 0$, is not uniquely defined due to zero modes. The minimum of the P_- may not correspond to the minimum of the Hamiltonian P_+. We take the approximation in which the vacuum is defined as the state minimizing the projection (i.e. the reduction) of the Hamiltonian P_+ on the eigen subspace with zero value of the P_-.

This projection depends only on zero modes of fields (which remain independent dynamical variables on the LF after the above mentioned limiting transition at finite L) and looks like the Hamiltonian of lattice gauge field theory in (2+1)-dimensions (where the x^+ enters formally like Lorentz time).

To find hadron masses we need to construct the states with finite value of P_-, using the LF Fock space basis for nonzero modes over that vacuum.

We hope that finally Lorentz invariance can be restored in the limit of removing regularization.

2 QCD Hamiltonian in coordinates close to LF coordinates

Let us start with the QCD Lagrangian in coordinates, $y^\mu = (y^0, y^1, y^2, y^3)$, approximating the LF coordinates:

$$y^0 = x^+ + \frac{\eta^2}{2} x^-, \quad y^3 = x^-, \quad y^\perp = x^\perp. \qquad (1)$$

The limiting transition to the theory on the LF corresponds to the limit $\eta \to 0$ of the parameter η. In these coordinates we have the following Lagrangian density:

$$L(y) = Tr\{F_{03}^2(y) + \sum_{k=1,2} \left(2\, F_{0k}(y)\, F_{3k}(y) + \eta^2 F_{0k}^2(y)\right) -$$

$$-F_{12}^2(y)\} + i\sqrt{2}\, \psi_+^+(y)\, D_0\, \psi_+(y) + \frac{i\,\eta^2}{\sqrt{2}}\, \psi_-^+(y)\, D_0\, \psi_-(y) +$$

$$+ i\sqrt{2}\, \psi_-^+(y)\, D_3\, \psi_-(y) + i\, \psi_-^+(y)\, (D_\perp - m)\, \psi_+(y) +$$

$$+ i\, \psi_+^+(y)\, (D_\perp + m)\, \psi_-(y), \tag{2}$$

where $F_{\mu\nu}(y) = \partial_\mu A_\nu(y) - \partial_\nu A_\mu(y) - ig[A_\mu(y),\, A_\nu(y)]$, the $A_\mu(y)$ are gluon fields, $D_\mu = \partial_\mu - ig A_\mu(y)$, $D_\perp = \sum_{k=1,2} \sigma_k D_k$, the σ_k being the Pauli matrices, the g is coupling constant, m is the quark mass, $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ is the quark bispinor field.

We introduce the lattice in $y^1 = x^1$, $y^2 = x^2$, with lattice spacing a, and in the y^0 the lattice with the spacing a_0 (further it will tend to zero in the corresponding transfer matrix in y^0). Also we take $|y^3| \leqslant L$ with periodic boundary conditions for fields in y^3. The $A_3(y)$ and quark fields can be related to lattice sites, but for the transverse gluon fields we define new special variables in the form of the following complex $N \times N$ matrices (for SU(N) gauge model):

$$M_\alpha(y) = \left(I + iga_\alpha \tilde{A}_\alpha(y)\right) U_\alpha(y)\,, \quad \alpha = 0,\, 1,\, 2,$$

$$a_1 = a_2 = a\,, \tag{3}$$

where the $U_\alpha(y)$ and $\tilde{A}_\alpha(y)$ will describe zero and nonzero modes correspondingly. The $U_\alpha(y)$ are unitary $N \times N$ matrices, related to lattice links $(y - a_\alpha e_\alpha, y)$ with the e_α being the unit vector along y^α axis. The $\tilde{A}_\alpha(y)$ are the Hermitian $N \times N$ matrices, related to corresponding sites.

We define the gauge transformation law as follows:

$$\tilde{A}_\alpha(y) \rightarrow \Omega(y)\tilde{A}_\alpha(y)\,\Omega^+(y)\,,$$
$$U_\alpha(y) \rightarrow \Omega(y)U_\alpha(y)\,\Omega^+(y - a_\alpha e_\alpha)\,, \qquad (4)$$

where the $\Omega(y)$ is the matrix of gauge transformation. For the $M_\alpha(y)$ the transformation is

$$M_\alpha(y) \rightarrow \Omega(y)M_\alpha(y)\,\Omega^+(y - a_\alpha e_\alpha).$$

Let us define the operator D_3 by the following equalities:

$$D_3\tilde{A}_\alpha(y) = \partial_3\tilde{A}_\alpha(y) - ig\left[A_3(y), \tilde{A}_\alpha(y)\right],$$
$$D_3U_\alpha(y) = \partial_3U_\alpha(y) - igA_3(y)U_\alpha(y) + igU_\alpha(y)A_3(y - a_\alpha e_\alpha),$$
$$D_3M_\alpha(y) = \partial_3M_\alpha(y) - igA_3(y)M_\alpha(y) + igM_\alpha(y)A_3(y - a_\alpha e_\alpha),$$
$$D_3\psi(y) = \left(\partial_3 - igA_3(y)\right)\psi(y). \tag{5}$$

Now we reduce the definition of the matrix $U_\alpha(y)$ by the following gauge invariant condition:

$$D_3 U_\alpha(y) = 0, \tag{6}$$

and for the matrices $\tilde{A}_\alpha(y)$ we require the exclusion from them the part satisfying $D_3 \tilde{A}_\alpha(y) = 0$. In the gauge $A_3 = 0$ these conditions are nothing but the separation of zero and nonzero Fourier modes of matrices $M_\alpha(y)$ in y^3. Analogous definitions can be taken for quark fields.

If we require for $U_\alpha(y)$ in the limit $a, a_0 \to 0$ in the gauge $A_3 = 0$ that

$$U_\alpha(y) \to \exp\Big(iga_\alpha A_{\alpha 0}(y)\Big) \to I + iga_\alpha A_{\alpha\, 0}(y)\,,$$

where the $A_{\alpha 0}(y)$ is zero mode of the field $A_\alpha(y)$ of the original theory, while for the $\tilde{A}_\alpha(y)$ the coincidence with nonzero mode part of the $A_\alpha(y)$, we get in any gauge for matrices $M_\alpha(y)$ the following correspondence:

$$M_\alpha(y) \to I + iga_\alpha A_\alpha(y) + O\Big((a_\alpha g)^2\Big). \qquad (7)$$

This allows to define the lattice analog of field strength tensor $F_{\mu\nu}(y)$:

$$G_{12}(y) = -\frac{1}{ga^2} \left(M_1(y)M_2(y-ae_1) - M_2(y)M_1(y-ae_2) \right),$$

$$G_{0k}(y) = -\frac{1}{gaa_0} \left(M_k(y)M_0(y-ae_k) - M_0(y)M_k(y-a_0e_0) \right),$$

$$k = 1, 2,$$

$$G_{3\alpha}(y) = \frac{1}{ga_\alpha} D_3 M_\alpha(y). \tag{8}$$

At $a, a_0 \to 0$ we get $G_{\mu\nu}(y) \to iF_{\mu\nu}(y)$.

We have the following transformation law under gauge transformations:

$$G_{3\alpha}(y) \rightarrow \Omega(y)\, G_{3\alpha}(y)\, \Omega^+(y - a_\alpha e_\alpha),$$
$$G_{\alpha\beta}(y) \rightarrow \Omega(y)\, G_{\alpha\beta}(y)\, \Omega^+(y - a_\alpha e_\alpha - a_\beta e_\beta). \quad (9)$$

Further, we can introduce the gauge invariant analog of the cutoff in p_3, using the cutoff in the eigen values q_3 of the operator D_3: $|q_3| \leqslant \Lambda$.

The action is:

$$
S(\eta) = \sum_{y^{\perp},\, y^0} \int_{-L}^{L} dy^3 \; a^2 a_0 \Bigg\{ Tr\Big(G_{03}^+(y)\, G_{03}(y) +
$$

$$
+\eta^2\, G_{0k}^+(y)\, G_{0k}(y) + G_{0k}^+(y)\, G_{3k}(y) + G_{3k}^+(y)\, G_{0k}(y) -
$$

$$
- G_{12}^+(y)\, G_{12}(y) \Big) - \frac{i}{a_0\sqrt{2}}\Big(\psi_+^+(y) M_0(y)\, \psi_+(y - a_0 e_0) - h.c. \Big) -
$$

$$
- \frac{i\eta^2}{2\sqrt{2}\, a_0}\Big(\psi_-^+(y) M_0(y)\psi_-(y - a_0 e_0) - h.c. \Big) +
$$

$$
+ i\sqrt{2}\, \psi_-^+(y) D_3(y)\psi_-(y) - \frac{i}{2a}\Big(\psi_-^+(y) M_k(y)\, \sigma_k \psi_+(y - a e_k) +
$$

$$
+ \psi_+^+(y) M_k(y)\, \sigma_k \psi_-(y - a e_k) - h.c. \Big) -
$$

$$
- im\Big(\psi_-^+(y)\, \psi_+(y) - \psi_+^+(y)\, \psi_-(y) \Big) \Bigg\}. \qquad (10)
$$

In the following we take the abbreviated way of writing the arguments of our fields: for the field in the integration (or summation) point y we do not write the argument, and for the field in the translated point $y \pm a_\alpha e_\alpha$ we write only the displacement, for example:
$f(y) = f, \quad f(y \pm a_\alpha e_\alpha) = f(\pm a_\alpha).$

To "freeze" the limiting transition to the LF for zero modes and keep them as independent dynamical variables we fix the parameter η at some finite value η_0 in corresponding terms of the action:

$$S(\eta, \eta_0) = S(\eta) + 2L(\eta_0^2 - \eta^2) \sum_{y^\perp, y^0} \left\{ \frac{1}{g^2 a_0} \, Tr \left\{ \left(U_k U_0 (-a_k) - \right. \right.$$

$$\left. - U_0 U_k(-a_0) \right) \left(U_0^+(-a_k) U_k^+ - U_k^+(-a_0) U_0^+ \right) \right\} -$$

$$- \frac{i}{2\sqrt{2} a_0} \left(\psi_{-,0}^+ \, U_0 \, \psi_{-,0}(y - a_0) - h.c. \right) \right\}. \qquad (11)$$

Here the $\psi_{-,0}$ is zero mode of quark field.

To construct the transfer matrix in y^0 and the Hamiltonian (M. Creutz. Gauge fixing, the transfer matrix, and confinement on a lattice. Phys. Rev. D. 1977) we apply the gauge $U_0 = I$, $A_3 = 0$.

Notice that $A_3 = 0$, in principle, is not possible for general class of fields with periodic boundary conditions in y^3. But in the case of QED(1+1) it was noticed that we can get good semiphenomenological description even with so reduced class of fields. Now we consider the same variant for simplicity.

The action takes the following form:

$$S_G(\eta) = \sum_{y^\perp, y^0} \int_{-L}^{L} dy^3\, a^2 a_0\; Tr\Bigg\{ \left(\partial_3 \tilde{A}_0\right)^2 -$$

$$-\frac{i}{gaa_0}\, \partial_3 \tilde{A}_k \Bigg[\left(I + iga\,\tilde{A}_k\right)\left(I + iga_0 U_k \tilde{A}_0(-a_k)\, U_k^+\right) -$$

$$- \left(I + iga_0 \tilde{A}_0\right)\left(I + iga\tilde{A}_k(-a_0)\right) U_k(-a_0)\, U_k^+ - h.c.\Bigg] +$$

$$+\frac{\eta^2}{g^2 a^2 a_0^2}\Bigg(\left(I + g^2 a_0^2 U_k \tilde{A}_0^2(-a_k) U_k^{-1}\right)\left(I + g^2 a^2 \tilde{A}_k^2\right) -$$

$$-\Bigg[U_k U_k^+(-a_0)\left(I - iga\tilde{A}_k(-a_0)\right)\left(I - iga\tilde{A}_0(-a_0)\right)$$

$$\left(I + iga\tilde{A}_k\right)\left(I + iga_0 U_k \tilde{A}_0(-a_k)\, U_k^+\right) + h.c.\Bigg] +$$

$$+\left(I + g^2 a^2 \tilde{A}_k^2(-a_0)\right)\left(I + g^2 a_0^2 \tilde{A}_0^2\right)\Bigg) - G_{12}^+ G_{12}\Bigg\} +$$

$$+ 2L(\eta_0^2 - \eta^2) \sum_{y^\perp, y^0}\Bigg\{ \frac{1}{g^2 a_0}\, Tr\Bigg\{ 2 - \left(U_k U_k^+(-a_0) - h.c.\right)\Bigg\}\Bigg\}\Bigg\}. \quad (12)$$

The product $U_k U_k^+(-a_0)$, entering this expression, corresponds to the elementary translation in time on the lattice. This is used in the procedure of the construction of the transfer matrix and the Hamiltonian in the limit $a_0 \to 0$ by the method of the paper M. Creutz: Gauge fixing, the transfer matrix, and confinement on a lattice (Phys. Rev. D. 1977).

The gluon part of the Hamiltonian turns out to be:

$$H_G = \sum_{y^\perp, k} \left\{ \frac{g^2}{4L\eta_0^2} \left(\pi_k^a - \frac{1}{2} \int_{-L}^{L} dy^3 \left(f^{abc} \tilde{\Pi}_k^b \tilde{A}_k^c \right) \right)^2 + \right.$$

$$+ \frac{1}{2\,\eta^2 a^2} \int_{-L}^{L} dy^3 (\tilde{\Pi}_k^a - a^2 \partial_3 \tilde{A}_k^a)^2 +$$

$$+ \int_{-L}^{L} dy^3 \left(\tilde{\Pi}_k^a \, Tr \left[\lambda^a \left(\frac{\tilde{A}_0 - U_k \tilde{A}_0(-a_k) U_k^{-1}}{a} - \right. \right. \right.$$

$$\left. \left. \left. - ig \left[\tilde{A}_k, (\tilde{A}_0 + U_k \tilde{A}_0(-a_k) U_k^{-1}) \right] \right) \right] \right) -$$

$$\left. - \frac{a^2}{2} (\partial_3 \tilde{A}_0^a)^2 + a^2 \, Tr \left(G_{12}^+ G_{12} \right) \right\} + O(\eta^2). \qquad (13)$$

29

Here $\tilde{\Pi}_k^a$ and $\tilde{A}_k^a = Tr(\lambda^a \tilde{A}_k)$ are canonically conjugated pairs of variables, λ^a are the SU(N) analog of Gell-Mann matrices:

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f^{abc} \frac{\lambda^c}{2},$$

f^{abc} are SU(N) structure constants
$(a, b, c = 1, ..., N^2 - 1)$,

$$[U_{k'}(y'), \pi_k^a(y)] = \delta_{kk'} \delta_{y^\perp y^{\perp'}} \frac{\lambda^a}{2} U_k(y),$$

$$[\pi_k^a(y), \pi_{k'}^b(y')] = i\, \delta_{kk'} \delta_{y^\perp y^{\perp'}} f^{abc} \pi_k^c(y). \qquad (14)$$

In the part of the action depending on quark fields unitary matrices of gluon zero modes enter this part without derivatives in time. So we need no additonal lattice in time, and can construct this part of the Hamiltonian by usual canonical formalism. Let us normalize fermion variables as follows:

$$\chi = 2^{1/4}\psi_+ \,,$$

$$\xi = 2^{-1/4}\eta \left(\psi_- - \frac{1}{2L} \int_{-L}^{L} dy^3 \psi_- \right) \,,$$

$$\xi_0 = 2^{-1/4}\frac{a\,\eta_0}{\sqrt{2L}} \int_{-L}^{L} dy^3 \psi_- \,.$$

Then we get:

$$S_\psi = a^2 \sum_{y^\perp} \int dy^0 \int_{-L}^{L} dy^3 \Bigg\{ i\chi_+^+ D_0\chi + i\xi^+ D_0\xi +$$

$$+ \frac{i\eta}{a\sqrt{2L}\eta_0}\xi_0^+ D_0\xi + \frac{i\eta}{a\sqrt{2L}\eta_0}\xi^+ D_0\xi_0 + \frac{i}{2La^2}\xi_0^+ \partial_0\xi_0 +$$

$$+ \frac{2i}{\eta^2}\xi^+\partial_3\xi - \frac{i}{2a}\left(\left(\frac{\xi^+}{\eta} + \frac{\xi_0^+}{\sqrt{2L}\,\eta_0 a}\right) M_k\sigma_k\chi(-a_k) - h.c.\right) -$$

$$- \left(\chi^+ M_k\sigma_k\left(\frac{\xi(-a_k)}{\eta} + \frac{\xi_0(-a_k)}{\sqrt{2L}\,\eta_0 a}\right) - h.c.\right) -$$

$$- im\left(\left(\frac{\xi^+}{\eta} + \frac{\xi_0^+}{\sqrt{2L}\,\eta_0 a}\right)\chi - h.c.\right)\Bigg\}, \qquad (15)$$

and the corresponding part of the Hamiltonian takes the following form:

$$H_\psi = a^2 \sum_{y^\perp} \int_{-L}^{L} dy^3 \left\{ -g \left(\chi_+^+ \tilde{A}_0 \chi + \xi^+ \tilde{A}_0 \xi + \frac{\eta}{a\sqrt{2L}\eta_0} \xi_0^+ \tilde{A}_0 \xi + \right. \right.$$

$$+ \frac{\eta}{a\sqrt{2L}\eta_0} \xi^+ \tilde{A}_0 \xi_0 \Bigg) - \frac{i}{2a} \left(\left(\frac{\xi^+}{\eta} + \frac{\xi_0^+}{\sqrt{2L}\,\eta_0 a} \right) M_k \sigma_k \chi(-a_k) - \right.$$

$$\left. -h.c. \right) - \left(\chi^+ M_k \sigma_k \left(\frac{\xi(-a_k)}{\eta} + \frac{\xi_0(-a_k)}{\sqrt{2L}\,\eta_0 a} \right) - h.c. \right) -$$

$$\left. - im \left(\left(\frac{\xi^+}{\eta} + \frac{\xi_0^+}{\sqrt{2L}\,\eta_0 a} \right) \chi - h.c. \right) \right\}. \qquad (16)$$

Equating the variation of total Hamiltonian in the $\tilde{A}_0$ to zero, we obtain the following constraint:

$$a^2 \partial_3^2 \tilde{A}_0^a = Tr \left\{ \lambda^a \left(\frac{U_k^{-1}(a_k)\, \tilde{\Pi}_k(a_k)\, U_k(a_k) - \tilde{\Pi}_k}{a} - \right. \right.$$

$$\left. - i\frac{g}{2} \left[\, \tilde{A}_k,\, \tilde{\Pi}_k \right] - i\frac{g}{2}\, U_k^{-1}(a_k) \left[\, \tilde{A}_k,\, \tilde{\Pi}_k \right] U_k(a_k) \right) \right\} +$$

$$+ \frac{ga^2}{2} \left(\chi^+ \lambda^a \chi + \xi^+ \lambda^a \xi + \frac{\eta}{\sqrt{2L}\, a\, \eta_0} \left(\xi_0^+ \lambda^a \xi + h.c. \right) \right) . \quad (17)$$

This equation allows to express the $\tilde{A}_0$ through other variables. Now we can write the full Hamiltonian as follows :

$$
\begin{aligned}
H = \sum_{y^\perp, k} \Bigg\{ & \frac{g^2}{4L\eta_0^2} \left(\pi_k^a - \frac{1}{2} \int_{-L}^{L} dy^3 \left(f^{abc} \tilde{\Pi}_k^b \tilde{A}_k^c \right) \right)^2 + \\
& + a^2 \int_{-L}^{L} dy^3 \Bigg[\frac{1}{2\,\eta^2 a^4} (\tilde{\Pi}_k^a - a^2 \partial_3 \tilde{A}_k^a)^2 + \\
& + \frac{1}{2} \left(\partial_3 \tilde{A}_0^a \right)^2 + Tr \left(G_{12}^+ G_{12} \right) - \frac{2i}{\eta^2}\, \xi^+ \partial_3 \xi + \\
& + \frac{i}{2a} \left(\left(\frac{\xi^+}{\eta} + \frac{\xi_0^+}{\sqrt{2L}\,\eta_0 a} \right) M_k \sigma_k \chi(-a_k) - h.c. \right) + \\
& + \left(\chi^+ M_k \sigma_k \left(\frac{\xi(-a_k)}{\eta} + \frac{\xi_0(-a_k)}{\sqrt{2L}\,\eta_0 a} \right) - h.c. \right) + \\
& + im \left(\left(\frac{\xi^+}{\eta} + \frac{\xi_0^+}{\sqrt{2L}\,\eta_0 a} \right) \chi - h.c. \right) \Bigg] \Bigg\} + O(\eta^2), \quad (18)
\end{aligned}
$$

where the quantity $\partial_3 \tilde{A}_0^a$ must be defined by the constraint (17) obtained above.

Furthermore all fields must be expressed in terms of their Fourier modes in y^3 (which are true independent variables):

$$\tilde{A}^a_k = \frac{1}{a\sqrt{2L}} \sum_{n\neq 0} \frac{a^a_{n,k} + a^{a+}_{-n,k}}{\sqrt{2\,|p_n|}}\, e^{-ip_n y^3},$$

$$\tilde{\Pi}^a_k = \frac{-ia}{\sqrt{2L}} \sum_{n\neq 0} \frac{a^a_{n,k} - a^{a+}_{-n,k}}{\sqrt{2}}\, \sqrt{|p_n|}\, e^{-ip_n y^3},$$

$$\chi^i_r = \frac{1}{a\sqrt{2L}} \sum_{n} \chi^i_{nr}\, e^{-ip_n y^3}, \qquad \xi^i_r = \frac{1}{a\sqrt{2L}} \sum_{n\neq 0} \xi^i_{nr}\, e^{-ip_n y^3}, \quad (19)$$

$$p_n = \tfrac{\pi n}{L}\,, n = 0, \pm 1, \pm 2, ...$$

Nonzero canonical (anti)commutators are

$$\left[a^a_{n,\,k}(y) \,,\; a^{b\,+}_{n',\,k'}(y') \right] = \delta_{kk'}\,\delta_{ab}\,\delta_{nn'}\,\delta_{y^\perp y^{\perp\prime}},$$

$$\left\{ \chi^i_{nr}(y),\; \chi^{j\,+}_{n'r'}(y') \right\} = \left\{ \xi^i_{nr}(y),\; \xi^{j\,+}_{n'r'}(y') \right\} = \delta_{nn'}\,\delta_{ij}\,\delta_{rr'}\,\delta_{y^\perp y^{\perp\prime}}.$$

One has to add to these relations also the relations for gluon zero modes.

3   Limiting transition to the LF QCD Hamiltonian

To consider the LF limit $(\eta \rightarrow 0)$ for the obtained Hamiltonian let us notice that at fixed values of the parameters L and a one can decompose this Hamiltonian in powers of η as follows:

$$H = \frac{1}{\eta^2}H_0 + \frac{1}{\eta}H_1 + H_2 + O(\eta^2).$$

Having this decomposition one can construct the analog of stationary perturbation theory in η:

$$(H-E)f = 0, \ f = f_0 + \eta f_1 + \cdots , \ \ E = \frac{1}{\eta^2}E_0 + \frac{1}{\eta}E_1 + E_2 + O(\eta).$$

Notice that the f_0 correspond to states in the limit $\eta \rightarrow 0$, i.e. on the LF. If we require the finiteness of the energy E in the limit $\eta \rightarrow 0$, we must put $E_0 = E_1 = 0$.

In lowest and next two orders in η one has the following eq-ns:

$$H_0 f_0 = 0, \quad H_0 f_1 + H_1 f_0 = 0, \quad H_0 f_2 + H_1 f_1 + (H_2 - E_2) f_0 = 0.$$

Let us substitute into these eq-ns the explicit expressions for corresponding pieces of the Hamiltonian.

The H_0 can be written in terms of the Fourier modes as follows:

$$
H_0 = \sum_{y^\perp} \int_{-L}^{L} dy^3 \left\{ \frac{1}{2a^2} \left(\tilde{\Pi}_k^a - a^2 \partial_3 \tilde{A}_k^a \right)^2 - 2ia^2 \xi^+ \partial_3 \xi \right\} =
$$

$$
= 2 \sum_{y^\perp} \sum_{n>0} \left\{ |p_n| \left(\sum_a a_{-n,k}^{a\,+} a_{-n,k}^{a} + \sum_i \left(\xi_{-n}^{i+} \xi_{-n}^{i} + \xi_n^{i} \xi_n^{i+} \right) \right) \right\},
$$

where we throw out the constant.

Therefore the subspace of states $\{f_0\}$ can be defined by the following eq-ns:

$$
a_{-n,k}^{a} f_0 = \xi_{-n}^{i} f_0 = \xi_n^{i+} f_0 = 0 \,, \qquad n > 0. \qquad (23)
$$

So the $\{f_0\}$ play the role of "vacuum" w.r.t. these modes.

In the next order in η we get

$$f_1 = -(1 - \mathcal{P}_0)H_0^{-1}(1 - \mathcal{P}_0)H_1 f_0.$$

For the remaining eq-n it is sufficient to consider the projection on the subspace $\{f_0\}$. Denoting the projector by $\mathcal{P}_0$, we obtain the following equality:

$$\mathcal{P}_0 \left(H_2 - H_1(1 - \mathcal{P}_0)H_0^{-1}(1 - \mathcal{P}_0)H_1 \right) f_0 = E_2 f_0.$$

The equation of this form can be considered as the eigenvalue problem for the LF Hamiltonian because the E_2 define the eigenvalues in the limit $\eta \to 0$. Thus we can take for the LF Hamiltonian the following expression:

$$P_+ = \mathcal{P}_0\left(H_2 - H_1(1 - \mathcal{P}_0)H_0^{-1}(1 - \mathcal{P}_0)H_1\right)\mathcal{P}_0.$$

In this expression the dependence on variables $a^a_{-n,k}$, ξ^i_{-n}, ξ^{i+}_n $(n > 0)$ can be eliminated by using the eq-ns

$$a^a_{-n,k}f_0 = \xi^i_{-n}f_0 = \xi^{i+}_n f_0 = 0\,, \qquad n > 0$$

(because $\mathcal{P}_0$ is the projector on $\{f_0\}$).

In result we obtain the following expression depending only on operators $a_{nk}^a, a_{nk}^{a^+}, b_{nr}^i, b_{nr}^{i^+}, d_{nr}^i, d_{nr}^{i^+}, (n > 0), U_k, \pi_k^a, \chi_0, \xi_0$:

$$P_+ = \sum_{x^\perp} \int_{-L}^{L} dx^- \left\{ \frac{g^2}{8L^2\eta_0^2} \left(\pi_k^a - \frac{i}{2} \sum_{n>0} \left(f^{abc} a_{nk}^{+b} a_{nk}^c \right) \right)^2 + \right.$$

$$+ \frac{a^2}{2} \left(F_{+-}^a \right)^2 + a^2 Tr \left(G_{12}^{LF^+} G_{12}^{LF} \right) +$$

$$\left. + \frac{g^2}{8L} \left(5N - 2 - \frac{4}{N} \right) \left(\sum_{n>0} \frac{1}{p_n} \right) Tr \left(A_k^{LF} A_k^{LF} \right) \right\} +$$

$$+ \sum_{x^\perp} \int_{-L}^{L} dx^- \left\{ \frac{i}{2\sqrt{2L}\,\eta_0} \left(\xi_0^+ \left(I + iga\, A_k^{LF} \right) U_k \sigma_k \chi(-a_k) + \right. \right.$$

$$\left. + \chi^+ \left(I + iga\, A_k^{LF} \right) U_k \sigma_k \xi_0(-a_k) - h.c. \right) +$$

$$+ \frac{ima}{2\sqrt{2L}\,\eta_0} \left(\left[\xi_0^+, \chi \right] - \left[\chi^+, \xi_0 \right] \right) -$$

$$- \frac{i}{8} \left(\chi^+(-a_{k'})\, \sigma_{k'}\, U_{k'}^{-1} \left(I - iga\, A_{k'}^{LF} \right) - \right.$$

$$- \chi^+(a_{k'}) \left(I + iga\, A_{k'}^{LF}(a_{k'}) \right) U_{k'}(a_{k'})\, \sigma_{k'} + 2ma\chi^+ \right)$$

$$\partial_-^{-1} \left(\left(I + iga\, A_k^{LF} \right) U_k \sigma_k \chi(-a_k) - \right.$$

$$- \sigma_k U_k^{-1}(a_k) \left(I - iga\, A_k^{LF}(a_k) \right) \chi(a_k) + 2ma\chi \right) +$$

$$+ \left(\frac{g^2 \left(N - \frac{1}{N} \right)}{16La^2} \sum_{m>0} \frac{1}{p_m} \right) \left(\sum_{n \neq m} \frac{\chi_n^+ \chi_n}{p_{m-n}} + \right.$$

$$\left. \left. + \left(\sum_{n>m} + \sum_{n<-m} \right) \frac{\chi_n^+ \chi_n}{p_n} \right) \right\}, \tag{24}$$

where

$$F^a_{+-} = Tr\left(\lambda^a F_{+-}\right),$$

$$F_{+-} = \frac{1}{a}\left(A^{LF}_k - \left(U^{-1}_k A^{LF}_k U_k\right)(a_k)\right) -$$

$$-\frac{ig}{2}\partial^{-1}_-\left(\left[\partial_- A^{LF}_k, A^{LF}_k\right] +\right.$$

$$\left.+\left[\partial_-\left(U^{-1}_k A^{LF}_k U_k\right), U^{-1}_k A^{LF}_k U_k\right](a_k) - \chi^+ \lambda^a \chi \frac{\lambda^a}{2}\right), (25)$$

$$G^{LF}_{12}(x) = -\frac{1}{ga^2}\left(\left(I + iga\, A^{LF}_1\right)U_1\right.$$

$$\left(I + iga\, A^{LF}_2(-a_1)\right)U_2(-a_1) - \left(1 \leftrightarrow 2\right)\right), \quad (26)$$

$$A_k^{LF}(x) = \frac{1}{a\sqrt{2L}} \sum_{n>0} \left(\frac{a_{nk}^a(x^\perp)e^{-ip_n x^-}}{\sqrt{2p_n}} + h.c. \right) \frac{\lambda^a}{2}, \quad (27)$$

$$\chi_r^i(x) = \frac{1}{a\sqrt{2L}} \sum_{n>0} \left(b_{nr}^i(x^\perp)\, e^{-ip_n x^-} + d_{nr}^{i+}(x^\perp)\, e^{ip_n x^-} \right) +$$

$$+ \frac{\chi_0}{a\sqrt{2L}} \,. \quad (28)$$

The divergent sums present in this expression for the Hamiltonian must be regularized by the condition $p_- = p_n = \frac{\pi n}{L} \leqslant \Lambda$.

The momentum operator P_- in terms of variables a^a_{nk}, $a^{a^+}_{nk}$, b^i_{nr}, $b^{i^+}_{nr}$, d^i_{nr}, $d^{i^+}_{nr}$ has the following form:

$$P_- = \sum_{x^\perp} \sum_{n>0} p_n \left(a^{a^+}_{nk} a^a_{nk} + b^{i^+}_{nr} b^i_{nr} + d^{i^+}_{nr} d^i_{nr} \right) \geqslant 0. \quad (29)$$

The operators a^a_{nk}, $a^{a^+}_{nk}$, b^i_{nr}, $b^{i^+}_{nr}$, d^i_{nr}, $d^{i^+}_{nr}$ are the annihilation and creation operators in the Fock space with the "vacuum" corresponding to $P_- = 0$. Therefore the projection of the Hamiltonian P_+ onto the subspace with $P_- = 0$ can be easily found:

$$(P_+)_{\tilde{0}} = \sum_{x^\perp} \left(\frac{g^2}{4L\,\eta_0^2} \pi_k^a \pi_k^a + \frac{4L}{g^2 a^2} Re\, Tr\left(I - U_{12}\right) \right) +$$

$$+ \frac{i}{2a\,\eta_0} \left(\xi_0^+ U_k \sigma_k \chi_0(-a_k) + \chi_0^+ U_k \sigma_k \xi_0(-a_k) + \right.$$

$$\left. + ma\left[\xi_0^+, \chi_0\right] - h.c. \right). \tag{30}$$

This expression looks like a Hamiltonian of (2+1)-dimensional gauge theory with the lattice in the space and continuous time x^+. One can try to minimize this Hamiltonian to define the vacuum. Analogously one can try to define the states with finite momentum p_- using the creation operators in the abovementioned LF Fock space.

4 Conclusion

An attempt to construct LF Hamiltonian with vacuum parameters like condensates being taken into account is presented here for QCD. The way we choose looks as semiphenomenological approach suggested by previous detailed consideration of the problem in QED(1+1). This way defines the modification of the theory on the LF at finite value of the "LF infrared" regularization parameter L ($|x^-| \leq L$), assuming that this modification does correct the spectrum of mass, introducing vacuum effects in it's description. We relate that modification with zero modes of fields in x^-. These modes become in this way independent dynamical variables on the LF.

The regularization and the separation of zero modes from the other modes introduces Lorentz nonsymmetry into the formulation of the theory at finite L. To keep gauge invariance a generalization of the notion of zero mode is given in a gauge invariant way. Also we introduce gauge invariant cutoff in the LC momentum p_- (that becomes possible in lattice formulation in transversal to LC coordinates).

We remain with the questions about the possibility of the renormalization of this model in the limit of removing the regularization and the possibility to compare this model with usual formulation in Lorentz coordinates at least in perturbation theory in coupling constant. In this pertubative way we can try to check the restoration of Lorentz symmetry in the limit of the removing the regularization.

References

[1] *P. A. M. Dirac.* Forms of relativistic dynamics. Rev. Mod. Phys. 1949. V. 21. P. 392-398.

[2] *A. M. Annenkova, V. A. Franke, E. V. Prokhvatilov.* The solution of Schroedinger equation on the light front for Sine-Gordon model. Vestnik. Leningr. State. University. 1985. V. 4. P. 80-82.

[3] *S. J. Brodsky, H.-C. V. Pauli, S. S. Pinsky.* Quantum Chromodynamics and other Field Theories on the Light Cone. Phys. Reports. 1998. V. 301. P. 299-486.

[4] *V. A. Franke, Yu. V. Novozhilov, E. V. Prokhvatilov.* On The Light Cone Formulation of Classical Nonabelian Gauge Theory. Lett. Math. Phys. 1981. V. 5. P. 239-245.

[5] *V. A. Franke, Yu. V. Novozhilov, E. V. Prokhvatilov.* On The Light Cone Quantization of Nonabelian Gauge Theory. Lett. Math. Phys. 1981. V. 5. P. 437-444.

[6] *S. A. Paston, V. A. Franke.* Comparison of quantum field perturbation theory for the light front with the theory in Lorentz coordinates. Theor. Math. Phys. 112: 1117-1130, 1997, Teor. Mat. Fiz. 112: 399-416, 1997. hep-th/9901110.

[7] *S. A. Paston, E. V. Prokhvatilov, V. A. Franke.*
Theor. Math. Phys. 2002. V. 131. P. 516. hep-
th/0302016.

[8] *S. A. Paston, E. V. Prokhvatilov, V. A. Franke.*
On the construction of corrected light-front Hamil-
tonian for QED_2. hep-th/0011224.

[9] *E. V. Prokhvatilov, V. A. Franke.* Approxi-
mate description of QCD condensates in light cone
coordinates. Sov. J. Nucl. Phys. 47: 559, 1988,
Yad. Fiz. 47: 882-883, 1988.

[10] *M. Creutz.* Gauge fixing, the transfer matrix, and
confinement on a lattice. Phys. Rev. D. 1977. V. 15.
P. 1128-1136.