# Tunnel determinants from zeta functions. Instanton and bounces in quantum mechanics

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# Outline

# Tunnel effect through quantum mechanical instantons

- Euclidean action

$$S_E[x] = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} dx \left\{ \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + U[x(\tau)] \right\} \quad , \quad \tau \in (-\frac{T}{2}, \frac{T}{2})$$

- Classically degenerate ground states

$$\left.\frac{dU}{dx}\right|_{x=x^{(a)}} = 0 \ , \ U(x^{(a)}) = 0 \ , \ a = 1, 2, \cdots, N \quad \Rightarrow |x^{(a)}\rangle$$

- Tunnel effect path integral formula

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + U(x) \quad , \quad \langle x^{(a+1)}| \, e^{[-\frac{T}{\hbar}\hat{H}]} \, |x^{(a)}\rangle = N \int \mathcal{D}[x(\tau)] e^{-\frac{S_E[x]}{\hbar}}$$

- Instanton: finite Euclidean action classical trajectories:

$$\frac{dx}{d\tau} = \sqrt{2U(x)} \quad \equiv \quad \tau = \tau_0 + \int \frac{dx}{\sqrt{2mU(x)}}$$

$$\lim_{\tau\to-\infty} \bar{x}(\tau) = x^{(a)} \ , \ \lim_{\tau\to\infty} \bar{x}(\tau) = x^{(a+1)} \ , \ S[\bar{x}] = \int_{x^{(a)}}^{x^{(a+1)}} dx \sqrt{2U(x)} = S_0 < +\infty \quad .$$

# Steepest descent method: semiclassical approximation

- One-instanton contribution

$$\langle x^{(a+1)}|e^{-\frac{T}{\hbar}}|x^{(a)}\rangle \simeq N e^{-S(\bar{x})}\mathrm{Det}\ \mathbb{L}^{-\frac{1}{2}}\ (1+\mathcal{O}(\hbar))$$

$$\mathbb{L} = -\frac{d^2}{d\tau^2} + v^2 + V(\tau) \quad , \quad v^2 + V(\tau) = \frac{d^2U}{dx^2}[\bar{x}(\tau)] \ , \ v^2 = \frac{d^2U}{dx^2}[x^{(a)}] \ , \ \forall a$$

- Zero mode: quotient of determinants for the Polyakov-Coleman dilute instanton gas

$$\Psi_0(\tau) = \frac{1}{\sqrt{S_0}}\frac{d\bar{x}}{d\tau} \ , \ \mathbb{L}_0 = -\frac{d^2}{d\tau^2} + v^2 \quad , \quad K = \left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} \left|\frac{\mathrm{Det}\ \mathbb{L}_0}{\mathrm{Det}'\ \mathbb{L}}\right|^{\frac{1}{2}}$$

- Ray-Singer regularization of functional determinants

$$\frac{\mathrm{Det}\mathbb{L}}{\mathrm{Det}\mathbb{L}_0} = \exp\left[\frac{d\zeta_{\mathbb{L}_0}}{ds}(0) - \frac{d\zeta_{\mathbb{L}}}{ds}(0)\right]$$

# Heat function, Mellin's transform, zeta function

- The $\mathbb{L}_0$- and $\mathbb{L}$-heat traces

$$\begin{aligned} h_{\mathbb{L}_0}(\beta) &= \frac{1}{2}e^{-v^2\beta} + \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-(k^2+v^2)\beta} \\ h_{\mathbb{L}}(\beta) &= 1 + \sum_{j=1}^{N-1} e^{-\omega_j^2\beta} + s_N e^{-\omega_N^2\beta} + \int_{-\infty}^{\infty} dk\, \rho_{\mathbb{L}}(k) e^{-(k^2+v^2)\beta} \\ s_N &= \frac{1}{2} \text{ if } \omega_N^2 = v^2 \quad , \quad s_N = 1 \text{ if } \omega_N^2 < v^2 \end{aligned}$$

- Mellin transforms and spectral zeta functions:

$$\zeta_{\mathbb{L}_0}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\, \beta^{s-1} h_{\mathbb{L}_0}(\tau) \quad , \quad \zeta_{\mathbb{L}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\, \beta^{s-1} h_{\mathbb{L}}(\beta) \quad .$$

- Main tools: total phase shifts and spectral densities

$$\rho_{\mathbb{L}_0}(k) = \frac{T}{2\pi} \quad , \quad \rho_{\mathbb{L}}(k) = \frac{T}{2\pi} + \frac{1}{2\pi}\frac{d\delta}{dk}(k)$$

# The simple pendulum: $U(x) = mgl\left(1-\cos\sqrt{\frac{g}{l}}x\right)$

- $$S(z) = ml\int du\left(\frac{1}{2}\frac{dz}{du}\frac{dz}{du} + (1-\cos z)\right) \quad , \quad z = \sqrt{\frac{g}{l}}x \ , \ u = g\tau$$

- Instanton and instanton well

$$z^{(0)} = 0 \qquad , \qquad \bar{z}(u) = 4\arctan e^{u}$$

$$\mathbb{L}_0 = -\frac{d^2}{du^2} + 1 \qquad , \qquad \mathbb{L} = -\frac{d^2}{du^2} + 1 - \frac{2}{\cosh^2 u}$$

- Phase shifts, spectral density, heat and zeta functions

$$\begin{aligned}
\delta(k) &= 2\arctan\frac{1}{k} \quad , \quad \rho_{\mathbb{L}}(k) - \rho_{\mathbb{L}_0}(k) = -\frac{1}{\pi}\frac{1}{k^2+1} \\
h_{\mathbb{L}_0}(\beta) &= \frac{T}{\sqrt{4\pi\beta}}e^{-\beta} \quad , \quad h_{\mathbb{L}}(\beta) = \frac{T}{\sqrt{4\pi\beta}}e^{-\beta} - \mathrm{Erfc}[\sqrt{\beta}] \\
\zeta_{\mathbb{L}_0}(s) &= \frac{T}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \quad , \quad \zeta_{\mathbb{L}}(s) = \frac{T}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} - \frac{1}{\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \quad .
\end{aligned}$$

# The simple pendulum

- Zeta function derivatives

$$\begin{aligned}
\frac{d\zeta_{\mathbb{L}_0}}{ds}(s) &= \frac{l}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left[\psi(s-\tfrac{1}{2})-\psi(s)\right]\\
\frac{d\zeta_{\mathbb{L}}}{ds}(s)-\frac{d\zeta_{\mathbb{L}_0}}{ds}(s) &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\left[H_s-H_{s-\frac{1}{2}}\right]\\
\frac{d\zeta_{\mathbb{L}_0}}{ds}(\varepsilon)-\frac{d\zeta_{\mathbb{L}}}{ds}(\varepsilon) &= \gamma_E+\psi(\tfrac{1}{2})+\left[\tfrac{1}{3}\pi^2+\gamma_E^2+\psi(\tfrac{1}{2})\left(2\gamma_E+\psi(\tfrac{1}{2})\right)\right]\varepsilon+\mathcal{O}(\varepsilon^2)
\end{aligned}$$

- Quantum pendulum tunnel effect determinant

$$\gamma_E+\psi(\frac{1}{2})=\log\frac{1}{4}\quad\Rightarrow\quad K=\frac{\mathrm{Det}\,\mathbb{L}}{\mathrm{Det}\,\mathbb{L}_0}=\frac{1}{\omega^2}\exp\left[\frac{d\zeta_{\mathbb{L}_0}}{ds}(0)-\frac{d\zeta_{\mathbb{L}}}{ds}(0)\right]=\frac{1}{4\omega^2}\quad,\quad\omega^2=\frac{g}{l}$$

# The double well: $U(x) = \frac{m}{2a^4}\left(x^2 - a^2\right)^2$

- $$S(z) = ma \int du \left( \frac{1}{2}\frac{dz}{du}\frac{dz}{du} + \frac{1}{2}(z^2-1)^2 \right) \quad , \quad z = \frac{x}{a} \; , \; u = a\tau$$

- Instanton and instanton well

$$z^{(0)} = -1 \quad , \quad \bar{z}(u) = \tanh u$$
$$\mathbb{L}_0 = -\frac{d^2}{du^2} + 4 \quad , \quad \mathbb{L} = -\frac{d^2}{du^2} + 4 - \frac{6}{\cosh^2 u}$$

- Bound states, phase shifts, spectral density, heat and zeta functions

$$\begin{aligned}
\lambda_0 &= 0 \; , \; f_0(u) = \mathrm{sech}^2 u \quad , \quad \lambda_3 = 3 \; , \; f_3(u) = \tanh u \, \mathrm{sech}^2 u \\
\delta(k) &= -2\arctan\frac{3k}{2-k^2} \quad , \quad \rho_{\mathbb{L}}(k) - \rho_{\mathbb{L}_0}(k) = -\frac{1}{\pi}\left(\frac{1}{k^2+1} + \frac{2}{k^2+4}\right) \\
h_{\mathbb{L}_0}(\beta) &= \frac{T}{\sqrt{4\pi\beta}} e^{-4\beta} \quad , \quad h_{\mathbb{L}}(\beta) = \frac{T}{\sqrt{4\pi\beta}} e^{-4\beta} + e^{-3\beta}\mathrm{Erf}[\sqrt{\beta}] - \mathrm{Erfc}[2\sqrt{\beta}] \\
\zeta_{\mathbb{L}}(s) - \zeta_{\mathbb{L}_0}(s) &= \frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s)} \left[ \frac{2}{3^{s+\frac{1}{2}}} \, {}_2F_1[\tfrac{1}{2}, s+\tfrac{1}{2}, \tfrac{3}{2}; -\tfrac{1}{3}] - \frac{1}{4^s}\frac{1}{s} \right]
\end{aligned}$$

# The double well

- Zeta function derivatives

$$\frac{d\zeta_{\mathbb{L}}}{ds}(s) - \frac{d\zeta_{\mathbb{L}_0}}{ds}(s) =$$

$$= \quad \frac{1}{\sqrt{\pi}} \frac{\Gamma[s+\frac{1}{2}]}{\Gamma[s]} \Big[ -2\cdot 3^{-\frac{1}{2}-s} {}_2F_1[\tfrac{1}{2}, \tfrac{1}{2}+s, \tfrac{3}{2}; -\tfrac{1}{3}] \Big( \log 3 + \psi(s) - \psi(\tfrac{1}{2}+s) \Big) +$$

$$+ \quad \frac{4^{-s}}{s^2} + \frac{4^{-2}}{s} \Big( \log 4 + \psi(s) - \psi(s+\tfrac{1}{2}) \Big) + 2\cdot 3^{-\frac{1}{2}-s} {}_2F_1^{(0,1,0,0)}[\tfrac{1}{2}, \tfrac{1}{2}+s, \tfrac{3}{2}; -\tfrac{1}{3}] \Big]$$

- Taylor expansion around $s = 0$

$$\frac{d\zeta_{\mathbb{L}}}{ds}(\varepsilon) - \frac{d\zeta_{\mathbb{L}_0}}{ds}(\varepsilon) = \log 48 + \mathcal{O}(\varepsilon)$$

- Quantum double well tunnel effect determinant

$$K = \frac{1}{\omega^2} \frac{\mathrm{Det}\,\mathbb{L}}{\mathrm{Det}\,\mathbb{L}_0} = \exp\Big[ \frac{d\zeta_{\mathbb{L}_0}}{ds}(0) - \frac{d\zeta_{\mathbb{L}}}{ds}(0) \Big] = \frac{1}{48} \frac{1}{\omega^2} \quad , \quad \omega^2 = \frac{1}{a^2}$$

# The heat trace expansion and zero modes

- Heat-trace asymptotic expansion, zero mode, and meromorphic structure of the zeta function

$$h_{\mathbb{L}}(\beta) - h_{\mathbb{L}_0}(\beta) = \frac{e^{-\beta v^2}}{\sqrt{4\pi}} \sum_{n=1}^{\infty} c_n(\mathbb{L})\, \beta^{n-\frac{1}{2}} + \mathrm{Erf}\,(v\sqrt{\beta})$$

$$\zeta_{\mathbb{L}}(s) - \zeta_{\mathbb{L}_0}(s) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{v^2}\right)^s \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{c_n(\mathbb{L})}{v^{2n-1}} \frac{\Gamma(s+n-\frac{1}{2})}{\Gamma(s)} - \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \right\}$$

$$\frac{d\zeta_{\mathbb{L}}}{ds}(s) - \frac{d\zeta_{\mathbb{L}_0}}{ds}(s) = \frac{1}{\sqrt{\pi}} \frac{1}{v^{2s}} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{c_n(\mathbb{L})}{v^{2n-1}} \frac{\Gamma(s+n-\frac{1}{2})}{\Gamma(s)} \left( \frac{1}{s} + H_{n+s-\frac{3}{2}} - H_s + \log \frac{1}{v^2} \right) - \right.$$

$$\left. - \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \left( H_{s-\frac{1}{2}} - H_s + \log \frac{1}{v^2} \right) \right\} .$$

- Logarithm of the partition function

$$\frac{d\zeta_{\mathbb{L}}}{ds}(\varepsilon) - \frac{d\zeta_{\mathbb{L}_0}}{ds}(\varepsilon) = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} c_n(\mathbb{L}) \frac{\Gamma(n-\frac{1}{2})}{v^{2n-1}} + \log(4v^2) + \mathcal{O}^2(\varepsilon)$$

- Asymptotic formulae for tunnel effect determinants between classically degenerate vacua

$$K = \frac{\det \mathbb{L}}{\det \mathbb{L}_0} = \exp\left[ \frac{d\zeta_{\mathbb{L}_0}}{ds}(0) - \frac{d\zeta_{\mathbb{L}}}{ds}(0) \right] = \frac{1}{4v^2} \exp\left\{ -\frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{v^{2n-1}} c_n(\mathbb{L}) \right\}$$

$$K(N_t) = \frac{\det \mathbb{L}}{\det \mathbb{L}_0} = \exp\left[ \frac{d\zeta_{\mathbb{L}_0}}{ds}(0) - \frac{d\zeta_{\mathbb{L}}}{ds}(0) \right] = \frac{1}{4v^2} \exp\left\{ -\frac{1}{\sqrt{4\pi}} \sum_{n=1}^{N_t} \frac{\Gamma(n-\frac{1}{2})}{v^{2n-1}} c_n(\mathbb{L}) \right\}$$

# The Razavy potential: $U(z)=\frac{1}{4}(\sinh^2 z-1)^2$

- Improved modification of the GDW expansion

$$h_{\mathbb{L}}(\beta)-h_{\mathbb{L}_0}(\beta)=\frac{e^{-\beta v^2}}{\sqrt{4\pi}}\sum_{n=1}^{\infty}c_n(\mathbb{L})\,\beta^{n-\frac{1}{2}}+e^{-\beta v^2}\sum_{j=1}^{N}e^{\frac{j^2v^2}{N^2}\beta}\mathrm{Erf}\left(\frac{jv}{N}\sqrt{\beta}\right)\ .$$

- Minima, instantons, vacuum and instanton fluctuation operators

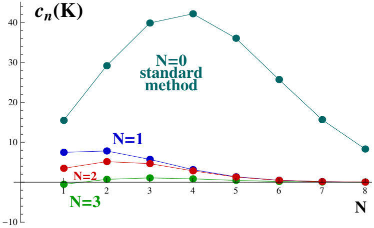
$$z^{(1)}=-\mathrm{arcsinh}\,1\ ,\ z^{(2)}=\mathrm{arcsinh}\,1\quad,\quad \bar{z}(u)=\mathrm{arctanh}\frac{\tanh u}{\sqrt{2}}$$

$$\mathbb{L}_0=-\frac{d^2}{du^2}+4\qquad,\qquad \mathbb{L}=-\frac{d^2}{du^2}+2+\frac{16}{(1+\mathrm{sech}^2u)^2}-\frac{14}{1+\mathrm{sech}^2u}$$

# The Razavy potential

- Zero mode and Seeley coefficients $c_n(\mathbb{L})$: $f_0(x) = \frac{4\sqrt{2}}{\sqrt{3\sqrt{2}\,\mathrm{arccosh}\,3 - 4(3+\cosh(2x)}}$

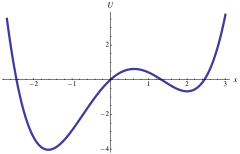
| $n$ | Seeley coefficients $N=0$ | $N=1$ | $N=2$ | $N=3$ |
|---|---|---|---|---|
| 1 | 15.4787 | 7.47870 | 3.47870 | -0.521297 |
| 2 | 29.1604 | 7.82708 | 5.16041 | 0.715970 |
| 3 | 39.8523 | 5.71900 | 4.65234 | 1.083616 |
| 4 | 42.1618 | 3.15228 | 2.84751 | 0.851307 |
| 5 | 36.0361 | 1.36104 | 1.29331 | 0.457301 |
| 6 | 25.7003 | 0.482076 | 0.469763 | 0.190385 |
| 7 | 15.6633 | 0.144365 | 0.142469 | 0.0647288 |
| 8 | 8.314336 | 0.037568 | 0.0373159 | 0.0224673 |
| 9 | 3.90348 | -0.0263172 | 0.00850317 | - |
| 10 | 1.64181 | 0.0018300 | - | - |

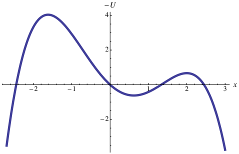


# False vacuum decay: $U(z) = 2z - \frac{3}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}$

- True and false vacua:

$$\underline{\text{true}} \quad z^{(1)} = -\frac{1+\sqrt{5}}{2} \qquad , \qquad \underline{\text{false}} \quad z^{(2)} = 2$$




- The bounce and the zero mode:

$$\lim_{u\to-\infty} \bar{z}(u) = 2 \ , \qquad \bar{z}(u) = \frac{2(10 - 10e^{\sqrt{5}u} + e^{2\sqrt{5}u})}{10 + 20e^{\sqrt{5}u} + e^{2\sqrt{5}u}} \ , \qquad \lim_{u\to+\infty} \bar{z}(u) = 2$$

$$\frac{d\bar{z}}{du} = \frac{60\sqrt{5}e^{\sqrt{5}u}(e^{\sqrt{5}u} - 10)}{(10 + 20e^{\sqrt{5}u} + e^{2\sqrt{5}u})^2}$$

# Negative mode and decay amplitude

- Fluctuation operators

$$\mathbb{L}_0 = -\frac{d^2}{du^2} + 5$$

$$\mathbb{L} = -\frac{d^2}{du^2} + 5 - \frac{2^3.3.5^2}{20 + 11\cosh\sqrt{5}u - 9\sinh\sqrt{5}u}\left(1 - \frac{6}{20 + 11\cosh\sqrt{5}u - 9\sinh\sqrt{5}u}\right)$$

- The false vacua life-time

$$\Gamma = \hbar\left|K\right| e^{-\int_{-\infty}^{\infty} du\left(\frac{dz}{du}\right)^2}$$

$$\left|K\right| = \frac{\left|\mathrm{Det}\,\mathbb{L}\right|}{\mathrm{Det}\,\mathbb{L}_0} = \frac{1}{4v^2}\left|e^{-\frac{1}{\sqrt{4\pi}}\sum_{n=1}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{v^{2n-1}}c_n(\mathbb{L})}\right|$$

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