New Approach to Duality-Invariant Nonlinear Electrodynamics

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Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

Summary and Outlook

Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

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Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

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The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

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Summary and Outlook

Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

Summary and Outlook

Outline

- Motivations
- The standard setting
- Formulation with bispinor auxiliary fields
- Alternative auxiliary field representation
- Examples
- Systems with higher derivatives
- $U(N)$ case
- Auxiliary superfields in self-dual $\mathcal{N}$=1 electrodynamics
- Summary and Outlook

Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

Summary and Outlook

Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

Summary and Outlook

Outline

Motivations

The standard setting

Formulation with bispinor auxiliary fields

Alternative auxiliary field representation

Examples

Systems with higher derivatives

$U(N)$ case

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

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$U(N)$ duality invariance: on-shell symmetry of a wide class of the nonlinear electrodynamics models including the renowned Born-Infeld theory

- Generalization of the free-case $O(2)$ symmetry between EOM and Bianchi ($F_{mn} = \partial_m A_n - \partial_n A_m,\ \tilde{F}_{mn} = \frac{1}{2}\epsilon_{mnpq}F^{pq}$):

$$\text{EOM} : \partial^m F_{mn} = 0 \qquad \Longleftrightarrow \qquad \text{Bianchi} : \partial^m \tilde{F}_{mn} = 0$$

$$\delta F_{mn} = -\omega \tilde{F}_{mn}, \qquad \delta \tilde{F}_{mn} = \omega F_{mn},$$

- In the nonlinear case: $P_{mn} = -2\frac{\partial L(F)}{\partial F^{mn}}$,

$$\text{EOM} : \partial^m P_{mn} = 0 \qquad \Longleftrightarrow \qquad \text{Bianchi} : \partial^m \tilde{F}_{mn} = 0\,,$$

$$\delta P_{mn} = -\omega \tilde{F}_{mn}\,, \qquad \delta \tilde{F}_{mn} = \omega P_{mn}$$

.

- Self-consistency condition (so called GZ condition) (M.Galliard, B.Zumino, 1981; G.Gibbond, D.Rasheed, 1995):

$$P\tilde{P} + F\tilde{F} = 0$$

- Now - rebirth of interest in the duality-invariant theories and their superextensions (R. Kallosh, G.Bossard, H. Nicolai, S.Ferrara, K. Stelle, ... , 2011, 2012,2013). The basic reason: generalized duality symmetry at quantum level can play the decisive role in proving the conjectured UV finiteness of $\mathcal{N} = 8, 4D$ supergravity!

To efficiently treat duality invariant theories and their supersymmetric extensions, new convenient general methods are urgently needed.

- A decade ago, a new general formulation of the duality invariant theories exploiting the tensorial (bispinor) auxiliary fields (E.I., B.Zupnik, 2001, 2002, 2004).
- The $U(N)$ duality is realized as the linear off-shell symmetry of the nonlinear interaction constructed out of the auxiliary fields. The GZ constraint is linearized in this formalism.
- In the $U(1)$ case the problem of restoring the nonlinear electrodynamics action by the auxiliary interaction is reduced to solving some algebraic equations. In the standard approach - differential equations are used.
- We realized that some methods recently invented for the systematic construction of various duality invariant systems in (G.Bossard & H.Nicolai, 2011; J.Carrasco, R.Kallosh & R.Roiban, 2012; W.Chemissany, R.Kallosh & T.Ortin, 2012) are in fact completely equivalent to our 10-years old approach just mentioned.
- This motivated us to return to the original formulation in order to see how the latest developments in the duality stuff can be ascribed into its framework (E.I. & B.Z., 2012,2013).

This will be the subjects of my talk.

The standard setting

- We make use of the bispinor formalism, e.g. :

$$F_{mn} \Rightarrow (F_{\alpha\beta}, \quad \bar{F}_{\dot{\alpha}\dot{\beta}}), \quad \varphi = F^{\alpha\beta}F_{\alpha\beta}\,, \quad \bar{\varphi} = \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}\,,$$

$$L(\varphi,\bar{\varphi}) = -\frac{1}{2}(\varphi+\bar{\varphi}) + L^{int}(\varphi,\bar{\varphi})$$

- E.O.M. and Bianchi:

$$\partial^{\dot{\beta}}_{\alpha}\bar{P}_{\dot{\alpha}\dot{\beta}}(F) - \partial^{\beta}_{\dot{\alpha}}P_{\alpha\beta}(F) = 0\,, \quad \partial^{\dot{\beta}}_{\alpha}\bar{F}_{\dot{\alpha}\dot{\beta}} - \partial^{\beta}_{\dot{\alpha}}F_{\alpha\beta} = 0\,, \quad P_{\alpha\beta} = i\frac{\partial L}{\partial F^{\alpha\beta}}$$

- $O(2)$ duality transformations

$$\delta_\omega F_{\alpha\beta} = \omega P_{\alpha\beta}, \qquad \delta_\omega P_{\alpha\beta} = -\omega F_{\alpha\beta},$$

- The self-consistency GZ constraint and the GZ representation for L:

$$F_{\alpha\beta}F^{\alpha\beta} + P_{\alpha\beta}P^{\alpha\beta} - \text{c.c} = 0 \quad \Leftrightarrow \quad \varphi - \bar{\varphi} - 4\,[\varphi(L_\varphi)^2 - \bar{\varphi}(L_{\bar{\varphi}})^2] = 0\,,$$

$$L = \frac{i}{2}(\bar{P}\bar{F} - PF) + I(\varphi,\bar{\varphi})\,, \quad \delta_\omega I(\varphi,\bar{\varphi}) = 0$$

- How to determine $I(\varphi,\bar{\varphi})$? Our approach with the auxiliary tensorial fields provides an answer.

Formulation with bispinor auxiliary fields

- Introduce the auxiliary unconstrained fields $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$ and write the extended Lagrangian in the (F, V)-representation as

$$\mathcal{L}(V,F) = \mathcal{L}_2(V,F) + E(\nu,\bar{\nu}), \quad \mathcal{L}_2(V,F) = \frac{1}{2}(\varphi+\bar{\varphi}) + \nu + \bar{\nu} - 2\,(V\cdot F + \bar{V}\cdot\bar{F}),$$

  with $\nu = V^2,\ \bar{\nu} = \bar{V}^2$. Here $\mathcal{L}_2(V,F)$ is the bilinear part only through which the Maxwell field strength enters the action and $E(\nu,\bar{\nu})$ is the nonlinear interaction involving only auxiliary fields.

- Dynamical equations of motion

$$\partial^{\dot{\beta}}_{\alpha}\bar{P}_{\dot{\alpha}\dot{\beta}}(V,F) - \partial^{\beta}_{\dot{\alpha}}P_{\alpha\beta}(V,F) = 0\,, \quad P_{\alpha\beta}(F,V) = i\frac{\partial\mathcal{L}(V,F)}{\partial F^{\alpha\beta}} = i(F_{\alpha\beta} - 2V_{\alpha\beta}),$$

  together with Bianchi identity, are covariant under $O(2)$ transformations

$$\delta V_{\alpha\beta} = -i\omega V_{\alpha\beta},\ \delta F_{\alpha\beta} = i\omega(F_{\alpha\beta} - 2V_{\alpha\beta}),\ \ \delta\nu = -2i\omega\nu$$

- Algebraic equations of motion for $V_{\alpha\beta}$,

$$F_{\alpha\beta} = V_{\alpha\beta} + \frac{1}{2}\frac{\partial E}{\partial V^{\alpha\beta}} = V_{\alpha\beta}(1 + E_{\nu})\,,$$

  is $O(2)$ covariant if and only if the proper constraint holds:

$$\nu E_{\nu} - \bar{\nu}E_{\bar{\nu}} = 0 \quad \Rightarrow \quad E(\nu,\bar{\nu}) = \mathcal{E}(a)\,,\ \ a := \nu\bar{\nu}$$

- The meaning of this constraint - $E(\nu, \bar{\nu})$ should be $O(2)$ invariant function of the auxiliary tensor variables

$$\delta_\omega E = 2i\omega(\bar{\nu} E_{\bar{\nu}} - \nu E_\nu) = 0$$

This is none other than the GZ constraint in the new setting:

$$F^2 + P^2 - \bar{F}^2 - \bar{P}^2 = 0 \quad \Leftrightarrow \quad \nu E_\nu - \bar{\nu} E_{\bar{\nu}} = 0$$

- The auxiliary equation can be now written as

$$F_{\alpha\beta} - V_{\alpha\beta} = V_{\alpha\beta} \bar{\nu}\, \mathcal{E}'$$

It and its conjugate serve to express $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ in terms of $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$:

$$V_{\alpha\beta}(F) = F_{\alpha\beta} G(\varphi, \bar{\varphi}), \quad G(\varphi, \bar{\varphi}) = \frac{1}{2} - L_\varphi = (1 + \bar{\nu}\mathcal{E}_a)^{-1}$$

After substituting these expressions back into $\mathcal{L}$ we obtain the corresponding selfdual $L(\varphi, \bar{\varphi})$

$$L^{sd}(\varphi, \bar{\varphi}) = \mathcal{L}(V(F), F) = -\frac{1}{2} \frac{(\varphi + \bar{\varphi})(1 - a\mathcal{E}_a^2) + 8a^2\mathcal{E}_a^3}{1 + a\mathcal{E}_a^2} + \mathcal{E}(a)$$

where a is related to $\varphi, \bar{\varphi}$ by the algebraic equation

$$(1 + a\mathcal{E}_a^2)^2 \varphi\bar{\varphi} = a[(\varphi + \bar{\varphi})\mathcal{E}_a + (1 - a\mathcal{E}_a^2)^2]^2$$

- The invariant GZ function: $I(\varphi, \bar{\varphi}) = \mathcal{E}(a) - 2a\mathcal{E}_a$, $a = \nu(\varphi, \bar{\varphi})\bar{\nu}(\varphi, \bar{\varphi})$.

To summarize, *all* $O(2)$ duality-symmetric systems of nonlinear electrodynamics without derivatives on the field strengths are parametrized by the $O(2)$ invariant off-shell interaction $\mathcal{E}(a)$ which is a function of the real quartic combination of the auxiliary fields. This universality is the basic advantage of the approach with tensorial auxiliary fields. The problem of constructing $O(2)$ duality-symmetric systems is reduced to choosing one or another specific $\mathcal{E}(a)$. As distinct from other existing approaches to solving the NGZ constraint, our approach uses the algebraic equations instead of the differential ones and automatically yields the Lagrangians $L^{sd}(\varphi, \bar{\varphi})$ which are analytic at $\varphi = \bar{\varphi} = 0$.

After passing to the tensorial notation, our basic auxiliary field equation (proposed ten years ago) precisely coincides with what was recently called "nonlinear twisted self-duality constraint" (G.Bossard & H.Nicolai, 2011; J.Carrasco, R.Kallosh & R.Roiban, 2012; W.Chemissany, R.Kallosh & T.Ortin, 2012) and $\mathcal{E}(a)$ with what is called there "duality invariant source of deformation".

Alternative auxiliary field representation

One of the advantages of our approach is the possibility to choose some other auxiliary field formulations which are sometimes simpler technically. It is convenient, e.g., to deal with the auxiliary complex variables μ and $\bar{\mu}$ related to ν and $\bar{\nu}$ by Legendre transformation

$$\mu := E_\nu, \; E - \nu E_\nu - \bar{\nu} E_{\bar{\nu}} := H(\mu, \bar{\mu}), \quad \nu = -H_\mu, \; E = H - \mu H_\mu - \bar{\mu} H_{\bar{\mu}}$$

Under the $O(2)$ duality $\delta_\omega \mu = 2i\omega\, \mu, \; \delta_\omega \bar{\mu} = -2i\omega\, \bar{\mu}$, so

$$H(\mu, \bar{\mu}) = I(b)\,, \qquad b = \mu\bar{\mu}$$

The basic relations of this formalism are

$$L^{sd}(\varphi, \bar{\varphi}) = -\frac{1}{2}\left(\varphi + \bar{\varphi} + 4bI_b\right) \frac{1-b}{1+b} + I(b), \quad (b+1)^2\, \varphi\bar{\varphi} = b\left[\varphi + \bar{\varphi} - I_b(b-1)^2\right]^2$$

These relations can be reproduced by eliminating μ and $\bar{\mu}$ as independent scalar auxiliary fields from the off-shell Lagrangian

$$\tilde{L}(\varphi, \mu) = \frac{\varphi(\mu - 1)}{2(1+\mu)} + \frac{\bar{\varphi}(\bar{\mu} - 1)}{2(1+\bar{\mu})} + I(\mu\bar{\mu}).$$

Yet exists a combined tensor - scalar auxiliary field “master” formulation which enables to establish relations between different equivalent off-shell descriptions of the same duality-invariant system:

$$\mathcal{L}(F, V, \mu) = \frac{1}{2}(F^2 + \bar{F}^2) - 2(VF) - (\bar{V}\bar{F}) + V^2(1+\mu) + (1+\bar{\mu})\bar{V}^2 + I(\mu\bar{\mu})$$

Examples

I. Born - Infeld. This model has a more simple description in the μ (or b) representation:

$$I^{BI}(b) = \frac{2b}{b-1}, \qquad I^{BI}_b = -\frac{2}{(b-1)^2}$$

The equation for computing b becomes quadratic:

$$\varphi\bar{\varphi}\, b^2 + [2\varphi\bar{\varphi} - (\varphi + \bar{\varphi} + 2)^2]\, b + \varphi\bar{\varphi} = 0 \qquad \Rightarrow$$

$$b = \frac{4\varphi\bar{\varphi}}{[2(1+Q) + \varphi + \bar{\varphi}]^2}, \quad Q(\varphi) = \sqrt{1 + \varphi + \bar{\varphi} + (1/4)(\varphi - \bar{\varphi})^2}$$

After substituting this into the general formula for $L^{cd}(\varphi, \bar{\varphi})$ the standard BI Lagrangian is recovered

$$L^{BI}(\varphi, \bar{\varphi}) = 1 - \sqrt{1 + \varphi + \bar{\varphi} + (1/4)(\varphi - \bar{\varphi})^2}\,.$$

II. The simplest interaction (SI) model. This model is the simplest example of the auxiliary interaction generating the non-polynomial self-dual electromagnetic Lagrangian. The relevant interaction in both ν and μ representations are linear functions

$$\mathcal{E}^{SI}(a) = \frac{1}{2}\,a\,, \quad I^{SI}(b) = -2b\,, \quad a = \nu\bar{\nu}, \;\; b = \mu\bar{\mu}.$$

Despite such a simple off-shell form of the auxiliary interaction, it is difficult to find a closed on-shell form of the nonlinear Lagrangian $L^{SI}(\varphi, \bar{\varphi})$, since the algebraic equations relating a (or b) to $\varphi, \bar{\varphi}$ are of the 5-th order. E.g.,

$$(b+1)^2\,\varphi\bar{\varphi} = b\,[\varphi + \bar{\varphi} + 2(b-1)^2]^2\,.$$

Nevertheless, it is straightforward to solve these equations as infinite series in b and then to restore $L^{SI}(\varphi, \bar{\varphi})$ to any order. Up to 10-th order in $F, \bar{F}$:

$$L_{sd} = -\frac{1}{2}(\varphi + \bar{\varphi}) + e_1\varphi\bar{\varphi} - e_1^2(\varphi^2\bar{\varphi} + \varphi\bar{\varphi}^2) + e_1^3(\varphi^3\bar{\varphi} + \varphi\bar{\varphi}^3)$$

$$+4e_1^3\varphi^2\bar{\varphi}^2 - e_1^4(\varphi^4\bar{\varphi} + \varphi\bar{\varphi}^4) - 10e_1^4(\varphi^3\bar{\varphi}^2 + \varphi^2\bar{\varphi}^3) + O(F^{12}).$$

Here $e_1 = \frac{1}{2}$.

Systems with higher derivatives

The nonlinear electromagnetic Lagrangians with higher derivatives are functions of the variables

$$F, \quad \partial_m F, \quad \partial_m \partial_n F, \quad \partial_m \partial_n \partial_r F \ldots$$

and their complex conjugates. The higher-derivative Lagrangians in the explicit form involve various scalar combinations of these variables

$$F^2, \;\; (\partial^m F \partial_m F), \;\; (\partial^m F^2 \partial_m F^2), \quad (F\Box^N F), \ldots .$$

It is known that the higher-derivative generalizations of the duality-invariant Lagrangians contain all orders of derivatives of $F_{\alpha\beta}$ and $F_{\dot\alpha\dot\beta}$ (G.Bossard & H.Nicolai, 2011).

In our formulation the generalized self-dual Lagrangian is

$$\mathcal{L}(F, V, \partial V) = \mathcal{L}_2 + \mathcal{E}(V, \partial V)$$

where $\mathcal{L}_2$ is the same "free" bilinear part as before and $\mathcal{E}(V, \partial V)$ is the $O(2)$ invariant self-interaction which can involve now any Lorentz invariant combinations of the tensorial auxiliary fields and their derivatives. In order to avoid non-localities and ghosts, it is reasonable to assume that $\mathcal{E}(V, \partial V)$ contains no terms bilinear in $V, \bar{V}$, i.e. that the extra derivatives appear only at the interaction level.

The equations of motion for this Lagrangian contain the Lagrange derivative of $\mathcal{E}$

$$\partial^{\alpha}_{\dot\beta}(F-2V)_{\alpha\beta}+\partial^{\dot\alpha}_{\beta}(\bar F-2\bar V)_{\dot\alpha\dot\beta}=0\,,\quad F_{\alpha\beta}=V_{\alpha\beta}+\frac{1}{2}\frac{\Delta\mathcal{E}}{\Delta V^{\alpha\beta}}$$

This set of equations together with the Bianchi identity for $F, \bar F$ is covariant under the $O(2)$ duality transformations, provided that $\mathcal{E}(V,\partial V)$ is $O(2)$ invariant,

$$\delta_\omega\,\mathcal{E}(V,\partial V)=0$$

An analog of GZ condition is the vanishing of the integral

$$\int d^4x[P^2(F,V)+F^2-\bar P^2(F,V)-\bar F^2]=0\,,\;P_{\alpha\beta}=i\frac{\Delta\mathcal{E}}{\Delta F^{\alpha\beta}}=i(F-2V)_{\alpha\beta}\,,$$

and this again amounts to the $O(2)$ invariance of $\mathcal{E}(V,\partial V)$.

Due to the property that the derivatives appear only in the interaction, one can solve the auxiliary field equations for $V_{\alpha\beta}, \bar V_{\dot\alpha\dot\beta}$ by recursions, like in the case without derivatives, and to finally obtain $L(F,\partial F)$ as a series expansion to any order in derivatives and the field strengths.

Some examples

As the first example we consider

$$\mathcal{E}_{(2)} = \frac{1}{2}\nu\bar{\nu} + c\partial^m\nu\partial_m\bar{\nu}, \quad \frac{\Delta\mathcal{E}_{(2)}}{\Delta V^{\alpha\beta}} = 2V_{\alpha\beta}[1 + \frac{1}{2}\bar{\nu} - c\Box\bar{\nu}]$$

The auxiliary field equation:

$$F_{\alpha\beta} = V_{\alpha\beta}(1 + \frac{1}{2}\bar{\nu} - c\Box\bar{\nu})$$

The perturbative solution:

$$\begin{aligned}
V^{(1)}_{\alpha\beta} &= F_{\alpha\beta}\,, \\
V^{(3)}_{\alpha\beta} &= -F_{\alpha\beta}(\frac{1}{2}\bar{\varphi} - c\Box\bar{\varphi}), \\
V^{(5)}_{\alpha\beta} &= F_{\alpha\beta}\{\frac{1}{2}\bar{\varphi}(\frac{1}{2}\bar{\varphi} - c\Box\bar{\varphi}) + \varphi(\frac{1}{2}\bar{\varphi} - c\Box\bar{\varphi}) - c(\Box\bar{\varphi})(\frac{1}{2}\bar{\varphi} - c\Box\bar{\varphi}) \\
&\quad - 2c\Box[\bar{\varphi}(\frac{1}{2}\varphi - c\Box\varphi)]\}, \qquad \text{etc}\,.
\end{aligned}$$

$$\begin{aligned}
L(F, \partial^N F) &= -\frac{1}{2}(\varphi + \bar{\varphi}) + \frac{1}{2}\varphi\bar{\varphi} - \frac{1}{4}\varphi^2\bar{\varphi} - \frac{1}{4}\varphi\bar{\varphi}^2 + c\partial^m\varphi\partial_m\bar{\varphi} \\
&\quad + c\varphi\bar{\varphi}[(\Box\varphi) + (\Box\bar{\varphi})] - c^2\varphi(\Box\bar{\varphi})^2 - c^2\bar{\varphi}(\Box\varphi)^2 + O(F^8)
\end{aligned}$$

This model can be regarded as the “minimal” higher-derivative deformation of the SI model (the latter is recovered at $c = 0$).

As the second example, we consider

$$\mathcal{E}_{(4)} = \gamma(\partial^m V \partial^n V)(\partial_m \bar{V} \partial_n \bar{V}),$$

where γ is a coupling constant and brackets denote traces with respect to the $SL(2, C)$ indices.

The auxiliary field equation:

$$F_{\alpha\beta} = V_{\alpha\beta} - \gamma\partial^m \left[\partial^n V_{\alpha\beta}(\partial_m \bar{V} \cdot \partial_n \bar{V})\right]. \quad (1)$$

The perturbative solution:

$$V^{(1)}_{\alpha\beta} = F_{\alpha\beta}, \quad V^{(3)}_{\alpha\beta} = \gamma\partial^m \left[\partial^n F_{\alpha\beta}(\partial_m \bar{F} \cdot \partial_n \bar{F})\right], \quad \text{etc}$$

The Lagrangian in the F-representation involves higher derivatives, starting from the sixth order in fields

$$L^{(6)} = (V^{(3)} V^{(3)}) - 2\gamma[(V^{(3)}\partial^n\partial^m F)(\partial_m \bar{F} \partial_n \bar{F}) + (V^{(3)}\partial^m F)\partial^n(\partial_m \bar{F} \partial_n \bar{F})] + \text{c.c.}$$

$U(N)$ case

The nonlinear Lagrangian with N abelian gauge field strengths

$$L(F^k,\bar{F}^l) = -\frac{1}{2}[(F^kF^k) + (\bar{F}^k\bar{F}^k)] + L^{int}(\varphi^{kl},\bar{\varphi}^{kl})\,,$$
$$\varphi^{kl} = \varphi^{lk} = (F^kF^l)\,, \quad \bar{\varphi}^{kl} = (\bar{F}^k\bar{F}^l)\,,$$

is chosen to be $O(N)$ invariant off shell

$$\delta_\xi F^k_{\alpha\beta} = \xi^{kl}F^l_{\alpha\beta}\,, \quad \delta_\xi \bar{F}^k_{\dot\alpha\dot\beta} = \xi^{kl}\bar{F}^k_{\dot\alpha\dot\beta}\,, \quad \xi^{kl} = -\xi^{lk}\,.$$

The nonlinear equations of motion

$$E^k_{\alpha\dot\alpha} := \partial^{\dot\beta}_\alpha \bar{P}^k_{\dot\alpha\dot\beta}(F) - \partial^\beta_{\dot\alpha}P^k_{\alpha\beta}(F) = 0\,,\ P^k_{\alpha\beta}(F) = i\frac{\partial L}{\partial F^{k\alpha\beta}}\,,$$

together with Bianchi

$$B^k_{\alpha\dot\alpha} = \partial^{\dot\beta}_\alpha \bar{F}^k_{\dot\alpha\dot\beta} - \partial^\beta_{\dot\alpha}F^k_{\alpha\beta} = 0\,.$$

are on-shell covariant under the $U(N)$ duality transformations

$$\delta_\eta F^k_{\alpha\beta} = \eta^{kl}P^l_{\alpha\beta}\,, \quad \delta_\eta P^k_{\alpha\beta} = -\eta^{kl}F^l_{\alpha\beta}\,,\ \eta^{kl} = \eta^{lk}\,,$$

provided that the GZ consistency conditions hold:

$$(P^kP^l) + (F^kF^l) - \mathrm{c.c.} = 0\,, \quad (F^kP^l) - (F^lP^k) - \mathrm{c.c.} = 0\,.$$

(F, V) representation

$$\mathcal{L}(F^k, V^k) = \mathcal{L}_2(F^k, V^k) + E(\nu^{kl}, \bar{\nu}^{kl}), \quad \nu^{kl} = (V^k V^l), \quad \bar{\nu}^{kl} = (\bar{V}^k \bar{V}^l),$$
$$\mathcal{L}_2(F^k, V^k) = \frac{1}{2}[(F^k F^k) + (\bar{F}^k \bar{F}^k)] - 2[(F^k V^k) + (\bar{F}^k \bar{V}^k)] + (V^k V^k) + (\bar{V}^k \bar{V}^k).$$

The $U(N)/O(N)$ duality transformations are implemented as

$$\delta_\eta F^k_{\alpha\beta} = \eta^{kl} P^l_{\alpha\beta} = i\eta^{kl}(F^l - 2V^l)_{\alpha\beta}, \quad \delta_\eta P^k_{\alpha\beta} = -\eta^{kl} F^l_{\alpha\beta}.$$

For the whole set of equations of motion to be $U(N)$ duality invariant, $E(\nu^{kl}, \bar{\nu}^{kl})$ should be $U(N)$ invariant:

$$E(\nu^{kl}, \bar{\nu}^{kl}) \Rightarrow \mathcal{E}(A_1, \ldots, A_N), \qquad A_1 = \bar{\nu}^{kl}\nu^{lk}, \ldots$$

The algebraic equations of the $U(N)$ duality-invariant models are

$$(F^k - V^k)_{\alpha\beta} = \mathcal{E}^{kl} V^l_{\alpha\beta}, \quad (\bar{F}^k - \bar{V}^k)_{\dot{\alpha}\dot{\beta}} = \bar{\mathcal{E}}^{kl} \bar{V}^l_{\dot{\alpha}\dot{\beta}}, \quad \mathcal{E}^{kl} := \frac{\partial \mathcal{E}}{\partial \nu^{kl}}.$$

These equations of motion are equivalent to the general "nonlinear twisted self-duality constraints". Solving them, e.g., by recursions, we can restore the whole nonlinear $U(N)$ duality invariant action by the invariant interaction $\mathcal{E}(A_1, \ldots, A_N)$. The duality invariant actions with higher derivatives can be constructed by the $U(N)$ invariant interaction involving derivatives of the auxiliary tensorial fields, like in the $U(1)$ case.

Auxiliary superfields in self-dual $\mathcal{N}=1$ electrodynamics

- The superfield action of nonlinear $\mathcal{N}=1$ electrodynamics:

$$S(W) = \frac{1}{4}\int d^6\zeta\, W^2 + \frac{1}{4}\int d^6\bar{\zeta}\,\bar{W}^2 + \frac{1}{4}\int d^8 z\, W^2\bar{W}^2 \Lambda(w,\bar{w},y,\bar{y})\,,$$

$$w = \frac{1}{8}\bar{D}^2\bar{W}^2\,, \quad \bar{w} = \frac{1}{8}D^2 W^2\,, \qquad y \equiv D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

- Here, $W_\alpha(x,\theta_\alpha,\bar{\theta}_{\dot\beta}) = \frac{i}{2}(\sigma^m\bar{\sigma}^n)_\alpha^\beta F_{mn}\theta_\beta - \theta_\alpha D + \ldots$, is the spinor chiral $\mathcal{N}=1$ Maxwell superfield strength ($\bar{D}_{\dot\alpha}W_\alpha = 0$, $D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$).
- The on-shell $U(1)$ duality rotations and the $N=1$ analog of the GZ self-duality constraint read

$$\delta W_\alpha = \omega M_\alpha(W,\bar{W})\,, \quad \delta M_\alpha = -\omega W_\alpha \quad M_\alpha := -2i\frac{\delta S}{\delta W^\alpha}\,,$$

$$\mathrm{Im}\int d^6\zeta\,(W^2 + M^2) = 0\,.$$

- The $\mathcal{N}=1$, $U(1)$ duality is the symmetry between the superfield equations of motion and Bianchi identity (S.Kuzenko, S.Theisen, 2000, 2001):

$$D^\alpha M_\alpha - \bar{D}_{\dot\alpha}\bar{M}^{\dot\alpha} = 0 \quad \Leftrightarrow \quad D^\alpha W_\alpha - \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha} = 0\,.$$

How to supersymmetrize the bispinor formulation?

- The basic idea (S.Kuzenko, 2013; E.I., O.Lechtenfeld, B.Zupnik, 2013): to embed tensorial auxiliary fields into chiral auxiliary superfields

$$V_{\alpha\beta}(x) \;\Rightarrow\; U_\alpha(x,\theta,\bar\theta) = v_\alpha(x) + \theta^\beta V_{\alpha\beta}(x) + \ldots\,,\;\; \bar D_{\dot\gamma} U_\alpha(x,\theta,\bar\theta) = 0$$

(similarly - for the $\mathcal{N} = 2$ case).

- We substitute $S(W) \to S(W,U)$, with

$$S(W,U) = \int d^6\zeta \left(UW - \frac{1}{2}U^2 - \frac{1}{4}W^2 \right) + \text{c.c.} + \frac{1}{4}\int d^8 z\, U^2 \bar U^2\, E(u,\bar u,g,\bar g),$$

$$u = \frac{1}{8}\bar D^2 \bar U^2, \quad \bar u = \frac{1}{8} D^2 U^2, \quad g = D^\alpha U_\alpha\,.$$

- Duality-invariant $\mathcal{N} = 1$ systems amount to the special choice of the $U(1)$ invariant interaction

$$E_{inv} = \mathcal{F}(B,A,C) + \bar{\mathcal{F}}(\bar B,A,C)\,, \quad A := u\bar u\,,\; C := g\bar g\,,\; B := ug^2\,,\; \bar B := \bar u\bar g^2\,.$$

M -representation

$$S(W,U,M) = \int d^6\zeta \left(UW - \frac{1}{2}U^2 - \frac{1}{4}W^2 \right) + \text{c.c.} + S_{int}(W,U,M)\,,$$

$$S_{int}(W,U,M) = \frac{1}{4}\int d^8z \left[(U^2\bar{M} + \bar{U}^2 M) + M\bar{M}\,J\,(m,\bar{m}) \right].$$

Here $m = \frac{1}{8}\bar{D}^2\bar{M}$, $\bar{m} = \frac{1}{8}D^2 M$ and M is a complex general scalar $\mathcal{N} = 1$ superfield. The duality transformations are realized as

$$\delta M = 2i\omega M\,, \quad \delta\bar{M} = -2i\omega\bar{M}\,, \quad \delta\bar{m} = 2i\omega\bar{m}\,, \quad \delta m = 2i\omega m\,.$$

The duality invariant systems correspond to the choice

$$J(m,\bar{m}) = J_{inv}(B)\,, \quad B := m\bar{m}\,.$$

Example. $\mathcal{N} = 1$ Born-Infeld theory: $J^{(BI)}_{inv}(B) = \frac{2}{B-1}$. This should be compared with the standard $W, \bar{W}$ representation of the same theory

$$S^{(BI)}_{int} = \frac{1}{4}\int d^8z\, W^2\bar{W}^2\Lambda^{(BI)}(w,\bar{w})\,, \quad w = \frac{1}{8}\bar{D}^2\bar{W}^2\,,\ \bar{w} = \frac{1}{8}D^2W^2\,,$$

$$\Lambda^{(BI)}(w,\bar{w}) = \left[1 + \frac{1}{2}(w+\bar{w}) + \sqrt{1 + (w+\bar{w}) + \frac{1}{4}(w-\bar{w})^2} \right]^{-1}.$$

Summary and Outlook

- **All** duality invariant systems of nonlinear electrodynamics (including those with higher derivatives) admit an off-shell formulation with the auxiliary bispinor (tensorial) fields. These fields are fully unconstrained off shell, there is no need to express them through any second gauge potentials, etc.

- The full information about the given duality invariant system is encoded in the $O(2)$ invariant interaction function which depends only on the auxiliary fields (or also on their derivatives) and can be chosen **at will**. In many cases it looks much simpler compared to the final action written in terms of the Maxwell field strengths.

- The renowned nonlinear GZ constraint is **linearized** in the new formulation and becomes just the requirement of $O(2)$ invariance of the auxiliary interaction. The $O(2)$ (and in fact $U(N)$, E.I. & B.Zupnik, 2013) duality transformations are linearly realized off shell.

- The basic algebraic equations eliminating the auxiliary tensor fields are equivalent to the recently employed "**nonlinear twisted self-duality constraints**". In our approach this sort of conditions appear as **equations of motion** corresponding to the well defined off-shell Lagrangian.

- *Some recent lines of development*:

  (a) Extensions to $\mathcal{N}=1$, $\mathcal{N}=2$, ... supersymmetric duality systems, including the most interesting supersymmetric Born-Infeld theories, in both flat and supergravity backgrounds (S. Kuzenko, 2013; E.I., O.Lechtenfeld, B. Zupnik, 2013).

  The basic point of $\mathcal{N}=1$ extension is embedding of tensorial auxiliary fields into chiral auxiliary superfields:

$$V_{\alpha\beta}(x) \;\Rightarrow\; U_\alpha(x,\theta,\bar\theta) = v_\alpha(x) + \theta^\beta V_{\alpha\beta}(x) + \ldots, \quad \bar D_{\dot\gamma} U_\alpha = 0$$

  Similarly - for $\mathcal{N}=2$ case.

  (b) Adding, in a self-consistent way, scalar and other fields into the auxiliary tensorial field formulation: $U(1)$ duality group $\Rightarrow SL(2,R)$ (G. Gibbons & D. Rasheed, 1995). Now in progress (E.I. & B. Zupnik).

  One adds two scalar fields, axion and dilaton, $\mathcal{S}_1$ and $\mathcal{S}_2$, which support a nonlinear realization of $SL(2,R)$, and generalize the self-dual Lagrangian in the (F,V)-representation as

$$L^{sd}(F,V) \;\Rightarrow\; \tilde L^{sd}(F,V,\mathcal{S}) = L(\mathcal{S}_1,\mathcal{S}_2) - \frac{i}{2}\mathcal{S}_1(\varphi - \bar\varphi) + L^{sd}(\sqrt{\mathcal{S}_2}F, V)$$

  The corresponding equations of motion together with Bianchi identity exhibit $SL(2,R)$ duality invariance. Similarly, $U(N)$ duality can be extended to $Sp(2N,R)$ by coupling to the coset $Sp(2N,R)/U(N)$ fields.

(c) The formulation presented suggests a **new view** of the duality invariant systems: in both the classical and the quantum cases **not** to eliminate the tensorial auxiliary fields by their equations of motion, but to deal with the off-shell actions **at all steps**. In many cases, the interaction looks much simpler when the auxiliary fields are retained in the action.

In this connection, recall the off-shell superfield approach in supersymmetric theories, which in many cases radically facilitates the quantum calculations and unveils the intrinsic geometric properties of the corresponding theories without any need to pass on shell by eliminating the auxiliary fields. Also it is worthwhile to recall that the tensorial auxiliary fields have originally appeared just within an off-shell superfield formulation of $\mathcal{N} = 3$ supersymmetric Born-Infeld theory (E.I., B.Zupnik, 2001).

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