Quantum fields over bounded domains

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Introduction

Quantum field theory in bounded domains ⇒ Boundary effects

1. Vacuum energy and the Casimir effect: possible explanation for dark energy
2. Flat systems, quantum wires and quantum dots
3. Quantum Hall effect and boundary states
4. Atoms in cavities
5. Effects over the topology of the universe

Other effects

- Black holes, open strings, D-branes, etc.

Introduction

Quantum field theory in bounded domains $\Rightarrow$ Boundary effects

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Outline of the talk:

1. Define QFT over bounded domains using AIM formalism for QM over bounded domains : consistency lemma
2. Spectral characterisation for $D = 1$
3. The boundary renormalization group: equations and flow
4. The vacuum energy and the Casimir effect: atractive, repulsive or null

The free scalar field over $\mathbb{R} \times M \subset \mathbb{R}^{D+1}$

Classical dynamics

$$S(\phi) = \frac{1}{2}\int_{\mathbb{R}\times M} d^{D+1}x \,\left(\partial^\mu \phi^* \partial_\mu \phi + m^2 |\phi|^2\right) + \frac{1}{2}\int_{\mathbb{R}\times \partial M} d^D x \phi^* \partial_n \phi$$

$$-\frac{\partial^2 \phi}{\partial t^2} = \mathbf{L}\,\phi; \quad \mathbf{L} = -\Delta + m^2$$

• Canonical quantization and unitarity of QFT $\Rightarrow$ **L** selfadjoint and non negative

- **L** selfadjoint $\Rightarrow$ Asorey-Ibort-Marmo (AIM) first theorem: characterisation of the set $\mathcal{M}$ of selfadjoint extensions of **L**
- **L** non-negative: no negative eigenvalues $\Rightarrow$ the strong consistency condition: characterisation of the set $\mathcal{M}_F$ of non-negative selfadjoint extensions.

Selfadjointness and non-negativity of L: the sets $\mathcal{M}$ and $\mathcal{M}_F$

- Selfadjointness of $\mathbf{L}_U$ $\Rightarrow$ first AIM theorem: $\mathcal{M} = \mathcal{U}\left(L^2(\partial M, \mathbb{C}^N)\right)$

$$\mathcal{D}(\mathbf{L}_U) = \left\{\psi \in L^2(M, \mathbb{C}^N) \ / \ \varphi - i\dot{\varphi} = U(\varphi - i\dot{\varphi})\right\}$$

being $\varphi = \psi|_{\partial M}$ and $\dot{\varphi} = \partial_n \psi|_{\partial M}$.

- Non-negativity of $\mathbf{L}_\mathbf{U}$ for any size of ∂M $\Rightarrow$ strong consistency condition

$$\mathbf{L}_U \geq 0 \Leftrightarrow i\left\langle \varphi, \frac{\mathbb{I} - U}{\mathbb{I} + U}\varphi \right\rangle \geq 0 \quad \forall \varphi \in L^2(\partial M, \mathbb{C}).$$

$$\mathcal{M}_F \equiv \left\{ U \in \mathcal{U}\left[L^2(\partial M, \mathbb{C})\right] \mid \forall\ \lambda = e^{i\theta} \in \sigma(U),\ \tan\left(\frac{\theta}{2}\right) \geq 0 \right\}.$$

Asorey, Ibort and Marmo, 2005. Asorey and JMMC 2008. JMMC 2009, Asorey and JMMC 2013.

The case $D = 1$. $M = [0,\ L] \Rightarrow \mathcal{M} = U(2)$ and $\mathcal{M}_F \subset U(2)$

- Euler parametrisation of $U(2)$ ($\alpha \in [0, 2\pi]$, $\beta \in [-\pi/2, \pi/2]$ and $\mathbf{n} \in S^2$):

$$U = \mathrm{e}^{i\alpha}\left(\cos(\beta)\mathbb{I} + i\sin(\beta)\,\mathbf{n}\cdot\boldsymbol{\sigma}\right) \ \in U(2)$$

- The space $\mathcal{M}_F$

$$\mathcal{M}_F \equiv \{U(\alpha,\ \beta,\ \mathbf{n}) \in U(2)\,|\,0 \leq \alpha \pm \beta \leq \pi\}$$

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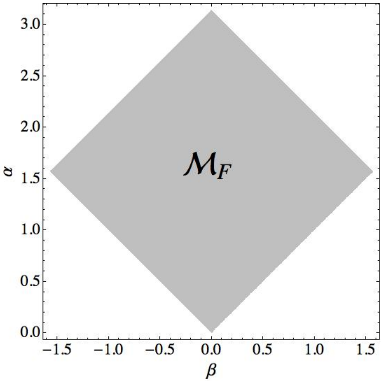


Figure: Dirichlet bc ($U = -\mathbb{I}$): $\alpha = \pi$, $\beta = 0$. Neumann bc ($U = \mathbb{I}$): $\alpha = \beta = 0$. Periodic bc ($U = \sigma_1$): $\alpha = \pi/2$, $n_1 = \pm 1$, $\beta = -n_1\alpha$.

$D = 1$. $M = [0, L]$. Characterisation of the spectrum $\mathbf{L_U} \in \mathcal{M}_F$

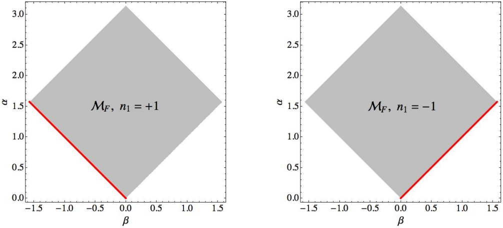
- Non-zero spectrum of $\mathbf{L_U}$: $\widehat{\sigma}(\mathbf{L_U})$. Spectral function

$$\begin{aligned} h_U(k;\alpha,\beta,n_1) &= 2ie^{i\alpha}\left[\left((k^2-1)\cos\beta + (k^2+1)\cos\alpha\right)\sin kL \right. \\ &- \left. 2k\sin\alpha\cos kL - 2k\, n_1 \sin\beta\right]. \end{aligned}$$

$h_U(k)$ does not characterise zero modes.

- $\mathbf{L_U} \in \mathcal{M}_F$ can only have a maximum of one constant zero mode.

$$\mathcal{M}_F^{(0)} = \{\mathbf{L_U} \in \mathcal{M}_F / \quad n_1 = \pm 1;\ \alpha \in [0, \pi/2];\ \beta = -n_1\alpha\}$$



Asorey and JMMC 2008, 2009, 2013. JMMC, K. Kirsten and M. Bordag 2013

The boundary renormalization group

• Massless scalar fields in $D+1$-dimensional space-time. Renormalization group of the boundary condition (boundary renormalization group)

$$U_\Lambda = \frac{\Lambda+1}{\Lambda-1}\,\frac{U+\frac{\Lambda-1}{\Lambda+1}}{U+\frac{\Lambda+1}{\Lambda-1}}$$

Boundary renormalization group flow:

$$\partial_\Lambda U_\Lambda = \frac{1}{2\Lambda}\left(\mathbb{I}-U_\Lambda^2\right) \Rightarrow U_\Lambda^\dagger \partial_\Lambda U_\Lambda = \frac{1}{2\Lambda}\left(U_\Lambda^\dagger - U_\Lambda\right)$$

Fixed points: $U = U^\dagger$. For 1+1 and $M=[0,\,L]$ $U=\pm\mathbb{I},\ \pm\mathbf{n}\cdot\boldsymbol{\sigma}$

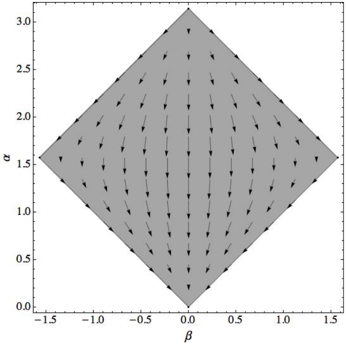
Boundary renormalization group. $1+1$, $M=[0,\ L]$

$$\mathbf{n} = (\sin(\theta)\cos(\gamma), \sin(\theta)\sin(\gamma), \cos(\theta))$$

- Boundary renormalization group flow

$$\alpha'(\Lambda) + \frac{1}{\Lambda}\sin(\alpha)\cos(\beta) = 0; \qquad \beta'(\Lambda) + \frac{1}{\Lambda}\cos(\alpha)\sin(\beta) = 0;$$

$$\theta'(\Lambda)^2 + \sin^2(\theta)\gamma'(\Lambda)^2 = 0$$



$$\theta'(\Lambda) = \sin(\theta)\gamma'(\Lambda) = 0$$

The Casimir effect between two parallel plates: arbitrary boundary condition

- Strong dependency with the dimension of the space-time

$$\frac{E_U^{(2n+1)}}{S} = \frac{4(-1)^n \Gamma\left(-\frac{2n+1}{2}\right) L_0^{2n+1}}{(4\pi)^{\frac{2n+3}{2}}(L^{2n+1} - L_0^{2n+1})} \int_0^\infty dk\; k^{2n+1} \left[\Phi_U(k;L) - \Phi_U(k;L_0)\right]$$

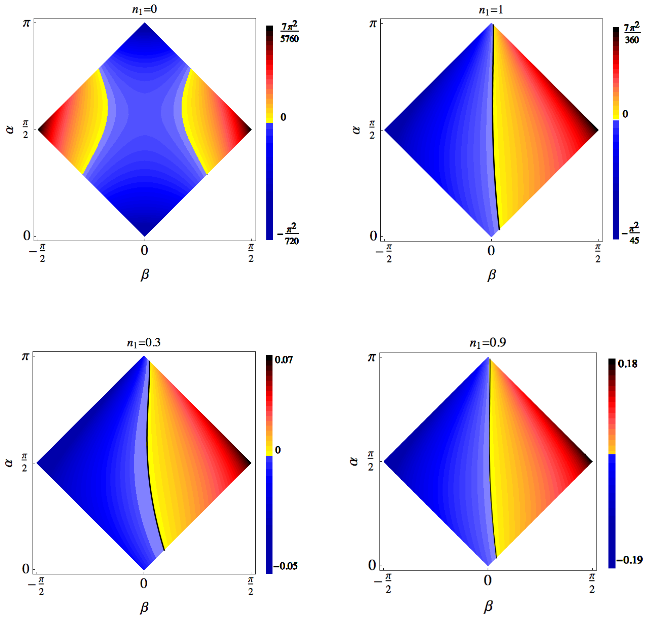
$$\frac{E_U^{(2n)}}{S} = -\frac{(4\pi)^{-n} L_0^{2n}}{\Gamma(n+1)(L^{2n} - L_0^{2n})} \int_0^\infty dk\, k^{2n} \left[\Phi_U(k;L) - \Phi_U(k;L_0)\right]$$

$$\Phi_U(k;L) \equiv L - \frac{d}{dk} \log\left(h_U^{(L)}(ik)\right)$$

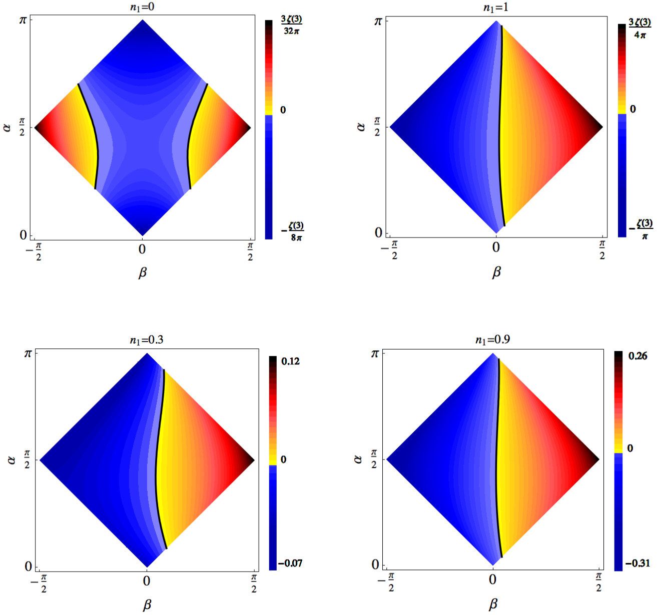
- Note that $E_U^{(D)}/S = c_U^{(D)}/L^D$

Asorey and JMMC 2013

$c_U^{(3)}$ as a function over $\mathcal{M}_F$



$c_U^{(2)}$ as a function over $\mathcal{M}_F$



Further comments and conclusions

- Conclusions
  - Characterisation of QFT-consistent boundary conditions
  - One-particle spectrum: characterisation of $\sigma(\mathbf{L}_U)$
  - Scale invariant boundary conditions: fixed points of the boundary RG flow
  - Casimirless, attractive, and repulsive boundary conditions
  - Minimum of the Casimir energy: periodic boundary conditions. Maximum: anti-periodic boundary conditions
  - Holomorphicity

Further comments and conclusions

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  - Dependence of the spectral zeta function and heat kernel asymptotic properties with the boundary conditions (JMMC, K. Kirsten, and M. Bordag 2013)
  - Special boundary conditions for graphene systems (Vassilevich *et al*)
  - Boundary conditions and topology change in cosmology (Asorey, Cavero-Pelaez and JMMC)
  - Lower bounds on the size of compact extra dimensions (Harbach and Hossenfelder, Phys.Lett.B 2005; work in progress by JMMC)
  - Arbitrary geometries
  - Scale invariant boundary conditions in string theories is an open question