Exact Cosmological Solutions In Induced Gravity Models

S.Yu. Vernov

Skobeltsyn Institute of Nuclear Physics,

Lomonosov Moscow State University

Saint-Petersburg, 4 October 2013

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I will speak about the reconstruction procedure and exact solutions in 2 models:

- E. Elizalde, E.O. Pozdeeva, and S.V., *Class. Quantum Grav.* **30** (2013) 035002, arXiv:1209.5957
- A.Yu. Kamenshchik, A. Tronconi, G. Venturi, and S.V., Phys. Rev. D **87** (2013) 063503, arXiv:1211.6272

FORMULATION OF NONLOCAL GRAVITY VIA SCALAR FIELDS

Action for nonlocal gravity (**the Elizalde talk**)

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\Box^{-1}R)\right)-2\Lambda\right]+\mathcal{L}_{\rm m}\right\}, \qquad (1)$$

where $\kappa^2 \equiv 8\pi/M_{\rm Pl}^2$, the Planck mass being $M_{\rm Pl} = G^{-1/2} = 1.2 \times 10^{19}$ GeV, $\Box$ is covariant d'Alembertian for a scalar field.
S. Deser and R. P. Woodard, *Phys. Rev. Lett.* **99** (2007) 111301, [arXiv:0706.2151].
This nonlocal model has a local scalar-tensor formulation.
S. Nojiri and S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821, [arXiv:0708.0924].

$$\tilde{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\eta)+\xi\right)-\xi\Box\eta-2\Lambda\right]+\mathcal{L}_{\rm m}\right\}. \qquad (2)$$

By varying action (2) over ξ, we get $\Box\psi = R$. Substituting $\psi = \Box^{-1}R$ into action (2), one reobtains nonlocal action (1).

By varying this action with respect to ξ and ψ, respectively, one obtains the field equations

$$\Box\psi = R\,, \qquad \Box\xi = f'(\psi)R\,,$$

where the prime denotes derivative with respect to ψ.
The Einstein equations

$$\frac{1}{2}g_{\mu\nu}\left[R\Psi + \partial_\rho\xi\partial^\rho\psi - 2(\Lambda + \Box\Psi)\right] - R_{\mu\nu}\Psi - \\ - \frac{1}{2}\left(\partial_\mu\xi\partial_\nu\psi + \partial_\mu\psi\partial_\nu\xi\right) + \nabla_\mu\partial_\nu\Psi = -\kappa^2 T_{\mathrm{m}\,\mu\nu}\,, \tag{3}$$

where $\Psi = 1 + f(\psi) + \xi$, $T_{\mathrm{m}\,\mu\nu}$ is the energy–momentum tensor of matter.
For the model, describing by the initial nonlocal action, a technique for choosing the distortion function so as to fit an arbitrary expansion history has been derived in *C. Deffayet and R.P. Woodard, JCAP* **0908** (2009) 023, [arXiv:0904.0961].
For the local formulation, a reconstruction procedure has been made in *T.S. Koivisto, Phys. Rev.* D **77** (2008) 123513, [arXiv:0803.3399] and *E. Elizalde, E.O. Pozdeeva, and S.Yu. V., Class. Quant. Grav.* **30** (2013) 035002, [arXiv:1209.5957].

In the spatially flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

for a perfect matter fluid, the Einstein equations are

$$3H^2\Psi = -\frac{1}{2}\dot{\xi}\dot{\psi} - 3H\dot{\Psi} + \Lambda + \kappa^2\rho_{\rm m}, \quad (4)$$

$$\left(2\dot{H} + 3H^2\right)\Psi = \frac{1}{2}\dot{\xi}\dot{\psi} - \ddot{\Psi} - 2H\dot{\Psi} + \Lambda - \kappa^2 P_{\rm m}, \quad (5)$$

$$\ddot{\xi} = -3H\dot{\xi} - 6\left(\dot{H} + 2H^2\right)f'(\psi), \quad (6)$$

$$\ddot{\psi} = -3H\dot{\psi} - 6\left(\dot{H} + 2H^2\right), \quad (7)$$

The Hubble parameter is $H = \dot{a}/a$.
The continuity equation for a perfect fluid with a EoS parameter $w_{\rm m}$ is

$$\dot{\rho}_{\rm m} = -3H(P_{\rm m} + \rho_{\rm m}) = -3H(1 + w_{\rm m})\rho_{\rm m}. \quad (8)$$

Adding up (4) and (5), we obtain the following second order linear differential equation for Ψ:

$$\ddot{\Psi} + 5H\dot{\Psi} + \left(2\dot{H} + 6H^2\right)\Psi - 2\Lambda + \kappa^2(P_{\rm m} - \rho_{\rm m}) = 0. \quad (9)$$

To reconstruct $f(\psi)$ and get a model with the exact solution for the given $H(t)$ and $w_{\rm m}(t)$ we can use the following algorithm:

- Assume the explicit form of $H(t)$ and $w_m(t)$.
- Solve **linear** equation (8) and get $\rho_{\rm m}(t)$.
- Solve **linear** equation (7) and get $\psi(t)$.
- Using $H(t)$, $w_{\rm m}(t)$, and $\rho_{\rm m}(t)$, solve **linear** equation (9) and get $\Psi(t)$.
- Substituting $\xi(t) = \Psi(t) - f(\psi) - 1$ into Eq. (6), we get a linear differential equation for $f(\psi)$:

$$\dot{\psi}^2 f''(\psi) - 12\left(\dot{H} + 2H^2\right) f'(\psi) = \ddot{\Psi} + 3H\dot{\Psi}\,. \qquad (10)$$

  To get (10) we also use *the inverse function* $t(\psi)$.
- Solve **linear** equation (10) and get the sought-for function $f(\psi)$.
- Substitute the obtained function $f(\psi)$ to Eq. (4) to check the existence of the solutions in the given form.

Note that equation (10) is a necessary condition that the model has the solutions in the given form.

E. Elizalde, E.O. Pozdeeva, and S.Yu.V.,
Phys. Rev. D **85** (2012) 044002, [arXiv:1110.5806]

Assuming that the Hubble parameter is a nonzero constant: $H = H_0$ we obtain that the model has de Sitter solutions if

$$f_1(\psi) = \frac{C_2}{4} e^{\psi/2} - \frac{\kappa^2 \rho_0}{3(1+3w_{\rm m})H_0^2} e^{3(w_{\rm m}+1)\psi/4}, \quad w_{\rm m} \neq -\frac{1}{3}.$$

$$\tilde{f}_1(\psi) = \left[\frac{\tilde{C}_2}{4} - \frac{\kappa^2 \rho_0}{12 H_0^2}\psi\right] e^{\psi/2}, \qquad w_{\rm m} = -\frac{1}{3},$$

$w_{\rm m}$ is a constant.
The function $f(\psi)$ can be determined up to a constant, because one can add it to $f(\psi)$ and subtract the same constant from ξ.

E. Elizalde, E.O. Pozdeeva, and S.Yu.V.,
Class. Quantum Grav. **30** (2013) 035002, [arXiv:1209.5957]
For $H = n/t$, we get that the model with

$$f(\psi) = \Lambda \tilde{f}_1 e^{\alpha_1 \psi} + \rho_0 \tilde{f}_2 \kappa^2 e^{\alpha_2 \psi} + C_1 \tilde{f}_3 e^{\alpha_3 \psi}, \qquad (11)$$

where $\tilde{f}_i$ and α_i are constants, has solutions with $H = n/t$.
C_1 is an arbitrary constant.
The constants are subject to the following conditions:

$$\tilde{f}_1 = \frac{t_0^2}{6n(1+n)}, \qquad \alpha_1 = \frac{1-3n}{3n(2n-1)},$$

$$\tilde{f}_2 = -\frac{t_0^{2-3n-3nw_{\rm m}}}{3n(n-2+3nw_{\rm m})}, \qquad \alpha_2 = \frac{(3n(1+w_{\rm m})-2)(3n-1)}{6n(2n-1)},$$

$$\tilde{f}_3 = \frac{(n-1)t_0^{-2n}}{2(2n-1)}, \qquad \alpha_3 = \frac{3n-1}{3(2n-1)}.$$

The method allows not only to get the suitable function $f(\psi)$, but also to obtain solutions in explicit form:

$$H(t)=\frac{n}{t},\qquad \rho_{\rm m}(t)=\rho_0 t^{-3n(w_{\rm m}+1)},$$

$$\psi(t)=-\frac{6n(2n-1)}{3n-1}\ln\left(\frac{t}{t_0}\right),\qquad \xi(t)=\Psi(t)-f(\psi)-1,$$

$$\Psi(t)=\begin{cases}\Theta+\dfrac{\Lambda t^2}{(n+1)(3n+1)}, & n\neq-\dfrac{1}{3},\\ \Theta+\dfrac{3}{2}\Lambda t^2\left(\ln(t)-\dfrac{3}{4}\right), & n=-\dfrac{1}{3},\end{cases}$$

$$\Theta\equiv C_1 t^{-2n}+C_2 t^{1-3n}-\frac{\rho_0\kappa^2(w_{\rm m}-1)t^{2-3(1+w_{\rm m})n}}{(3nw_{\rm m}-1)(n+3nw_{\rm m}-2)}.$$

Power-law solutions for the function

$$f=f_0 e^{\alpha\psi}.$$

have been considered in
E. Elizalde, E.O. Pozdeeva, S.Yu.V., Y.-l. Zhang ,
JCAP **1307** (2013) 034, arXiv:1302.4330.

Let us consider the model with the following action

$$S=\int d^4x\sqrt{-g}\left[U(\sigma)R-\frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu}-V(\sigma)\right],$$

In FLRW metric: $ds^2=\,-\,dt^2+a^2(t)\left(dx_1^2+dx_2^2+dx_3^2\right),$
we get the following equations:

$$6UH^2+6\dot{U}H=\frac{1}{2}\dot{\sigma}^2+V, \tag{12}$$

$$2U\left(2\dot{H}+3H^2\right)+4\dot{U}H+2\ddot{U}=\,-\,\frac{1}{2}\dot{\sigma}^2+V. \tag{13}$$

Combining Eqs. (12) and (13) we obtain:

$$4U\dot{H}-2\dot{U}H+2\ddot{U}+\dot{\sigma}^2=0. \tag{14}$$

This equation plays a key role in the reconstruction procedure.

For the case of induced gravity $U(\sigma) = \xi\sigma^2$ the reconstruction procedure has been proposed in
A.Yu. Kamenshchik, A. Tronconi, G. Venturi, *Reconstruction of scalar potentials in induced gravity and cosmology*,
Phys. Lett. B **702** (2011) 191–196, arXiv:1104.2125.
They got a lot of potential for different types of the Hubble behaviors.
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There are two main reasons to use the **superpotential** *method*:

- $U(\sigma)$ can be arbitrary function. For example, $U(\sigma) = \xi\sigma^2 + J$.
- $H(t)$ can be more complicated than $H = Y(\sigma)$.

The two methods supplement each other and together allow one to construct different cosmological models with some required properties.

SUPERPOTENTIAL METHOD

Let

$$H = Y(\sigma),$$
$$\dot{\sigma} = F(\sigma),$$

then Eq. (14) is

$$4U\dot{H} - 2\dot{U}H + 2\ddot{U} + \dot{\sigma}^2 = 0 \quad \Leftrightarrow$$

$$4UY_{,\sigma} + 2(F_{,\sigma} - Y)U_{,\sigma} + (2U_{,\sigma\sigma} + 1)F = 0. \tag{15}$$

The potential

$$V(\sigma) = 6UY^2 + 6U_{,\sigma}FY - \frac{1}{2}F^2. \tag{16}$$

Equation (15) contains three functions. If two of them are given, then the third one can be found as the solution of a linear differential equation. If $U(\sigma)$ and $F(\sigma)$ are given, then

$$Y(\sigma) = -\left(\int^{\sigma} \frac{2F_{,\tilde{\sigma}}U_{,\tilde{\sigma}} + (2U_{,\tilde{\sigma}\tilde{\sigma}} + 1)F}{4U^{3/2}} d\tilde{\sigma} + c_0\right)\sqrt{U} \qquad (17)$$

For given $Y(\sigma)$ and $U(\sigma)$, we obtain

$$F(\sigma) = \left[\int^{\sigma} \frac{U_{,\tilde{\sigma}}Y - 2UY_{,\tilde{\sigma}}}{U_{,\tilde{\sigma}}} e^{\Upsilon} d\tilde{\sigma} + \tilde{c}_0\right] e^{-\Upsilon(\sigma)}, \qquad (18)$$

where

$$\Upsilon(\sigma) \equiv \frac{1}{2}\int^{\sigma} \frac{2U_{,\tilde{\sigma}\tilde{\sigma}} + 1}{U_{,\tilde{\sigma}}} d\tilde{\sigma}.$$

The key point in this method is that the Hubble parameter is considered as a function of the scalar field.
The Hamilton–Jacobi formulation (superpotential method) has been proposed in the cosmological models with minimally coupling scalar field:
A.G. Muslimov, *Class. Quant. Grav.* **7** (1990) 231–237;
D.S. Salopek, J.R. Bond, *Phys. Rev.* D **42** (1990) 3936–3962;
and has been develop in:
I.Ya. Aref'eva, A.S. Koshelev, S.Yu.V.,
Theor. Math. Phys. **148** (2006) 895–909, astro-ph/0412619;
Phys. Rev. D **72** (2005) 064017, astro-ph/0507067;
D. Bazeia, C.B. Gomes, L. Losano, R. Menezes,
Phys. Lett. B **633** (2006) 415–419; astro-ph/0512197;
K. Skenderis, P.K. Townsend,
Phys. Rev. D **74** (2006) 125008, hep-th/0609056;
A.A. Andrianov, F. Cannata, A.Yu. Kamenshchik, and D. Regoli,
JCAP **0802** (2008) 015, arXiv:0711.4300;
A.V. Yurov, V.A. Yurov, S.V. Chervon, and M. Sami,
Theor. Math. Phys. **166** (2011) 259–269 ...

Models with $U(\sigma) = \xi\sigma^2 + J$

Let us consider models with non-minimally coupling scalar field with

$$U(\sigma) = \xi\sigma^2 + J,$$

where ξ and J are constant.
The superpotential method allows to get the potentials $V(\sigma)$ for models with

- de Sitter solutions,
- asymptotic de Sitter solutions,
- power-law solutions, that reproduce the cosmological evolution given by Einstein–Hilbert action plus a barotropic perfect fluid.

A.Yu. Kamenshchik, A. Tronconi, G. Venturi, and S.Yu.V.,
Phys. Rev. D **87** (2013) 063503, arXiv:1211.6272.

NON-MONOTONIC BEHAVIOR OF THE HUBBLE PARAMETER IN THE CASE OF INDUCED GRAVITY

The same evolution $\sigma(t)$ can lead to exactly solvable models with different potentials and different qualitative behavior of the Hubble parameter.

Let $U(\sigma) = \xi\sigma^2$.

Let us consider $Y(\sigma)$ as a quadratic polynomial:

$$Y(\sigma) = A_2\sigma^2 + A_1\sigma + A_0,$$

where A_k are constants.

The function $F(\sigma)$ **does not** depend on A_1:

$$F(\sigma) = \frac{4\xi}{8\xi + 1} A_0\sigma - \frac{4\xi}{16\xi + 1} A_2\sigma^3 + \tilde{c}_0\sigma^{-\frac{1+4\xi}{4\xi}}.$$

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When $\tilde{c}_0 = 0$, $F(\sigma)$ is a cubic polynomial and the equation $\dot{\sigma} = F(\sigma)$ has the following general solution (does not depend on A_1):

$$\sigma(t) = \pm\frac{\sqrt{(16\xi+1)A_0}}{\sqrt{(16\xi+1)A_0c_2e^{-\omega t} + (8\xi+1)A_2}}, \qquad (19)$$

where $\omega = 8\xi A_0/(8\xi+1)$, c_2 is an arbitrary integration constant.

POTENTIAL

The corresponding potential is the sixth degree polynomial, which has the following form (at $\xi = 1$):

$$\begin{aligned} V(\sigma) &= \frac{910}{289}A_2^2\sigma^6 + \frac{156}{17}A_1A_2\sigma^5 + \\ &+ \left(6A_1^2 + \frac{2236}{153}A_0A_2\right)\sigma^4 + \frac{52}{3}A_0A_1\sigma^3 + \frac{910}{81}A_0^2\sigma^2 . \end{aligned}$$

If $\omega > 0$, then

$$\lim_{t\to\infty}\sigma(t) = \pm\frac{(16\xi+1)\sqrt{A_0}}{(8\xi+1)\sqrt{A_2}} . \qquad (20)$$

In the case $\omega < 0$, the function $\sigma(t)$ tends to zero at late times. Hence, the Hubble parameter tends to a constant at late times for any case.

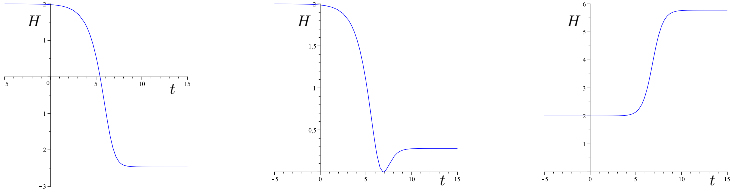

Figure : **The function $H(t)$ with $A_1 = -6$, $A_1 = -4$, and $A_1 = 0$ (from left to right). At all pictures we use $A_2 = 1$, $A_0 = 2$, and $c_2 = 100000$.**

The same function $\sigma(t)$ is associated with different behaviors of the Hubble parameter.
At $A_1 = -4$ we get a non-monotonic behavior of $H(t)$.

Conclusions

- A nonlocal gravity model with a function $f(\Box^{-1}R)$ has been considered. We have extended the reconstruction procedure for the scalar-tensor model, which is a local formulation of this model.
- It has been proved that this model has solutions with the given $H(t)$ only if the function f satisfies the second-order linear differential equation (10).
- For de Sitter and power-law solutions, f is a sum of exponential functions.
- Cosmological models with non-minimally coupling scalar fields has been considered. The superpotential method has been used for the reconstruction procedure.
- We have investigated a few models having a different de Sitter asymptotic behaviour in the past and in the future. Non-monotonic behaviour have been found.