CORRELATION FUNCTIONS OF THE XXZ HEISENBERG CHAIN IN THE IZING LIMIT

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Introduction

We shall consider the Heisenberg XXZ chain in two specific limits of the anysotropy parameter Δ. The $\Delta \to \infty$ limit is called the Izing limit, while $\Delta \to 0$ is the XX (free fermion) limit. The wave functions in the considered limits are expressed in terms of the symmetric Schur functions. We shall calculate the thermal correlators of the ferromagnetic string over the ground states in both limits

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) \equiv \frac{\langle \Psi_N(\boldsymbol{\theta}) \mid \bar{\Pi}_n \, e^{-\beta \widehat{H}} \, \bar{\Pi}_n \mid \Psi_N(\boldsymbol{\theta}) \rangle}{\langle \Psi_N(\boldsymbol{\theta}) \mid e^{-\beta \widehat{H}} \mid \Psi_N(\boldsymbol{\theta}) \rangle},$$

$$\bar{\Pi}_n = \sigma^+_M \sigma^-_M \dots \sigma^+_n \sigma^-_n$$

and study its low-temperature ($\beta \to \infty$) asymptotics when the number of the lattice sites are much greater than the number of particles $M \gg N$. The amplitudes of these asymptotics are related to the number of the boxed plane partitions (three dimensional Young diagrams) in a $N \times N \times (M - N - n)$ box.

XXZ SPIN CHAIN

• •• Hamiltonian of the XXZ model defined on a chain of "length" M+1 in absence of magnetic field has the form:

$$\widehat{H}_{XXZ} = -\frac{1}{2}\sum_{k=0}^{M}\left(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-} + \frac{\Delta}{2}\left(\sigma_{k+1}^{z}\sigma_{k}^{z} - 1\right)\right),$$

where $\Delta \in \mathbb{R}$ is internal anisotropy, and M+1 is even. The local spin operators σ_n^a are defined as tensor products:

$$\sigma_n^a \equiv \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \underbrace{\sigma^a}_{n^{\text{th}}} \otimes \cdots \otimes \mathbb{I},$$

where σ^a, $a = x, y, z$, are the Pauli matrices at n^{th} place, and $\sigma_n^{\pm} = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$.

The algebra of the spin operators is given by the commutation relations:

$$\left[\sigma_k^{+}, \sigma_l^{-}\right] = \delta_{kl}\,\sigma_l^{z}\,, \qquad \left[\sigma_k^{z}, \sigma_l^{\pm}\right] = \pm 2\,\delta_{kl}\,\sigma_l^{\pm}\,.$$

The spin operators act over the state-space $\mathfrak{H}_{M+1}$ given by the tensor product: $\mathfrak{H}_{M+1} = \bigotimes\limits_{k=0}^{M} \mathbb{C}^2$. The linear space $\mathbb{C}^2$ is spanned over the spin "up" and "down" states ($|\uparrow\rangle$ and $|\downarrow\rangle$, respectively):

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The space $\mathfrak{H}_{M+1}$ is spanned over of the state-vectors $\bigotimes\limits_{k=0}^{M} |s\rangle_k$, where s implies either $\uparrow$ or $\downarrow$. The periodic boundary conditions $\sigma_{k+(M+1)}^{\#} = \sigma_k^{\#}$ are imposed.

• • • XX model

The Hamiltonian is given by the limit $\Delta \to 0$ in the XXZ Hamiltonian:

$$\widehat{H}_{XX} \equiv -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-})\,.$$

• • • *Izing (Strong Anisotropy)* limit $\Delta \to -\infty$

Less studied limit of the XXZ model is the *Strong Anisotropy* (SA) limit $\Delta \to -\infty$. In this limit the system can be described by the effective Hamiltonian $\widehat{H}_{\mathsf{SA}}$:

$$\widehat{H}_{\mathsf{SA}} = -\frac{1}{2}\sum_{k=0}^{M}\mathcal{P}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-})\mathcal{P}\,,$$

where the projectors $\mathcal{P} \equiv \prod_{k=0}^{M} (1 - \widehat{q}_{k+1}\widehat{q}_k)$ cut out the states with the spin "down" states at any pair of nearest-neighboring sites. The projectors onto the spin "up" and "down" states are:

$$\check{q}_k \equiv \frac{1}{2}(\sigma_k^0 + \sigma_k^z), \quad \widehat{q}_k \equiv \frac{1}{2}(\sigma_k^0 - \sigma_k^z), \qquad k \in \mathcal{M},$$

Refs.: F. C. Alcaraz, R. Z. Bariev, *An exactly solvable constrained XXZ chain*, arXiv:cond-mat/9904042

N. I. Abarenkova, A. G. Pronko, *The temperature correlator in the absolutely anisotropic Heisenberg XXZ-magnet*, Teoret. Mat. Fiz., **131** (2002), 288–303.

The N-particle state-vectors of the XXZ model are represented in the form

$$\boxed{\,|\,\Psi_N(u_1,\dots,u_N)\rangle = \sum_{\{e_k(\boldsymbol{\mu})\}_{k\in\mathcal{M}}} \chi_{\boldsymbol{\mu}}^{XXZ}(u_1,u_2,\dots,u_N)\prod_{k=0}^{M}(\sigma_k^-)^{e_k}\,|\Uparrow\rangle\,,\,}$$

The sites with spin "down" states are labeled by the coordinates μ_i, $1 \le i \le N$. These coordinates form a strict partition $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, \dots, \mu_N)$, where $M \ge \mu_1 > \mu_2 > \dots > \mu_N \ge 0$. There is a correspondence between each partition and an appropriate sequence of zeros and unities of the form: $\left\{e_k \equiv e_k(\boldsymbol{\mu})\right\}_{k\in\mathcal{M}}$, where $e_k \equiv \delta_{k,\mu_n}$, $1 \le n \le N$. The condition $\sum\limits_{k=0}^{M} e_k = N$ is satisfied.

$$\boxed{|\Psi_N(u_1,\ldots,u_N)\rangle = \sum_{\{e_k(\boldsymbol{\mu})\}_{k\in\mathcal{M}}} \chi_{\boldsymbol{\mu}}^{XXZ}(u_1,u_2,\ldots,u_N) \prod_{k=0}^{M} (\sigma_k^-)^{e_k} \,|\Uparrow\rangle\,,}$$

The wave function is:

$$\chi_{\boldsymbol{\mu}}^{XXZ}(u_1,u_2,\ldots,u_N) = \sum_{S_{p_1,p_2,\ldots,p_N}} \mathcal{A}_S(u_1,u_2,\ldots,u_N)\, u_{p_1}^{2\mu_1} u_{p_2}^{2\mu_2} \ldots u_{p_N}^{2\mu_N}\,,$$

where summation goes over all elements of the group of permutations $S_{p_1,p_2,\ldots,p_N} \equiv S\begin{pmatrix} 1, & 2, & \ldots, & N \\ p_1, & p_2, & \ldots, & p_N \end{pmatrix}$. The amplitude:

$$\mathcal{A}_S(u_1,u_2,\ldots,u_N) \equiv \prod_{1\le j<i\le N} \frac{1-2\Delta u_{p_i}^2 + u_{p_i}^2 u_{p_j}^2}{u_{p_i}^2 - u_{p_j}^2}\,.$$

The state-vectors are the eigen-states of the XXZ Hamiltonian with corresponding eigen-values,

$$\widehat{H}_{XXZ}\,|\,\Psi_N(u_1,\ldots,u_N)\rangle = E_N\,|\,\Psi_N(u_1,\ldots,u_N)\rangle\,,$$

if and only if, the parameters u_l ($1 \leq l \leq N$) satisfy the *Bethe equations*:

$$\boxed{u_l^{2(M+1)} = (-1)^{N-1}\prod_{k=1}^{N}\frac{1-2\Delta u_l^2+u_l^2u_k^2}{1-2\Delta u_k^2+u_l^2u_k^2}\,.}$$

The corresponding eigen-energies are given by

$$\boxed{E_N \equiv E_N(u_1,\ldots,u_N) = -\frac{1}{2}\sum_{i=1}^{N}(u_i^2+u_i^{-2}-2\Delta)\,.}$$

Up to irrelevant pre-factor, the wave function at $\Delta = 0$ is equal to

$$\chi^{XX}_{\boldsymbol{\mu}}(u_1, u_2, \ldots, u_N) = \det(u_j^{2\mu_k})_{1\le j,k\le N} \prod_{1\le n<l\le N} (u_l^2 - u_n^2)^{-1},$$

and the Bethe equation with its solution are of the form:

$$u_j^{2(M+1)} = (-1)^{N-1}, \qquad u_j^2 = e^{i\frac{2\pi}{M+1}I_j}, \qquad 1 \le j \le N,$$

where I_j are integers or half-integers depending on whether N is odd or even: $M \ge I_1 > I_2 > \cdots > I_N \ge 0$. The eigen-energy is of the form:

$$E^{XX}_N(I_1, I_2, \ldots, I_N) = -\sum_{l=1}^{N} \cos\Big(\frac{2\pi I_l}{M+1}\Big).$$

The wave function in the Izing limit $\Delta \to -\infty$ takes the form (up to a pre-factor):

$$\chi^{\mathsf{SA}}_{\boldsymbol{\mu}}(u_1, u_2, \ldots, u_N) = \det(u_j^{2(\mu_k - N + k)})_{1 \le j,k \le N} \prod_{1 \le n < l \le N} (u_l^2 - u_n^2)^{-1},$$

where the spin "down" states form strict decreasing partition $\boldsymbol{\mu}$, i.e., $M \ge \mu_1 > \mu_2 > \ldots > \mu_N \ge 0$. The wave function is not equal to zero if and only if the elements μ_i, $1 \le i \le N$, satisfy the *exclusion condition*: $\mu_i > \mu_{i+1} + 1$. The Bethe equation with its solution are specialized as follows:

$$u_k^{2(M+1-N)} = (-1)^{N-1} \prod_{j=1}^{N} u_j^{-2}, \qquad u_k^2 = e^{i \frac{2\pi I_k - P}{M+1-N}}, \qquad 1 \le k \le N.$$

where I_j are integers or half-integers depending on N being odd or even, whereas $P \equiv \frac{2\pi}{M+1} \sum_{j=1}^{N} I_j$.

The eigen-energy has the form:

$$E_N^{SA}(I_1, I_2, \dots, I_N) = -\sum_{l=1}^{N} \cos\Big(\frac{2\pi I_k - P}{M+1-N}\Big),$$

where $M - N \geq I_1 > I_2 > \cdots > I_N \geq 0$.

• • • The wave functions of the model in the considered limits may be expressed through the *Schur functions*:

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) \equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\le j,k\le N}}{\det(x_j^{N-k})_{1\le j,k\le N}} = \det(x_j^{\lambda_k+N-k})_{1\le j,k\le N} \prod_{1\le n<l\le N} (x_l - x_n)^{-1},$$

where $\boldsymbol{\lambda}$ is $(\lambda_1, \lambda_2, \dots, \lambda_N)$ being N-tuple of non-increasing non-negative integers: $L \ge \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge 0$.

Any strict partition $M \ge \mu_1 > \mu_2 > \dots > \mu_N \ge 0$ and non-strict partition $M + 1 - N \ge \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge 0$ (denoted as $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, respectively) are connected by the equation $\lambda_j = \mu_j - N + j$, where $j = 1, \ldots, N$. In other terms, $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the strict partition $(N-1, N-2, \ldots, 1, 0)$.

So the wave function of the XX model may be represented as

$$\boxed{\chi_{\boldsymbol{\mu}}^{XX}(u_1, u_2, \ldots, u_N) = S_{\boldsymbol{\lambda}}(u_1^2, \ldots, u_N^2)\,.}$$

Any strict decreasing partition $\boldsymbol{\mu}$ respecting the exclusion condition $\mu_i > \mu_{i+1} + 1$ is related to the non-strict partition $\widetilde{\boldsymbol{\lambda}}$: $\widetilde{\boldsymbol{\lambda}} = \boldsymbol{\mu} - 2\boldsymbol{\delta}$, where $M + 2(1-N) \geq \widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots \geq \widetilde{\lambda}_N \geq 0$. Therefore, we obtain:

$$\boxed{\chi_{\boldsymbol{\mu}}^{\mathsf{SA}}(u_1, u_2, \ldots, u_N) = S_{\widetilde{\boldsymbol{\lambda}}}(u_1^2, \ldots, u_N^2)\,.}$$

The scalar products of the state-vectors in the both considered limits, and the certain expectation values we will calculate with the help of the Binet–Cauchy formula:

$$\boxed{\begin{aligned} & \sum_{\boldsymbol{\lambda}\subseteq\{L^N\}} S_{\boldsymbol{\lambda}}(x_1^2,\dots,x_N^2) S_{\boldsymbol{\lambda}}(y_1^2,\dots,y_N^2) \\ = \; & \det(T_{jk})_{1\le j,k\le N} \prod_{1\le k<j\le N} \left(y_j^2 - y_k^2\right)^{-1} \prod_{1\le m<l\le N} \left(x_l^2 - x_m^2\right)^{-1}, \end{aligned}}$$

where summation goes over all non-strict partitions $\boldsymbol{\lambda}$: $L \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge 0$. The entries T_{jk} take the form:

$$T_{jk} = \frac{1-(x_k y_j)^{2(N+L)}}{1-(x_k y_j)^2}.$$

CORRELATION FUNCTIONS

• • • We shall demonstrate the discussed approach to the calculation of correlation functions on the example of the survival probability of ferromagnetic string:

$$\boxed{\mathcal{T}(\boldsymbol{\theta}, n, \beta) \equiv \frac{\langle \Psi_N(\boldsymbol{\theta}) \mid \bar{\Pi}_n \, e^{-\beta \widehat{H}} \, \bar{\Pi}_n \mid \Psi_N(\boldsymbol{\theta}) \rangle}{\langle \Psi_N(\boldsymbol{\theta}) \mid e^{-\beta \widehat{H}} \mid \Psi_N(\boldsymbol{\theta}) \rangle},}$$

where $\check{q}_j \equiv \frac{\sigma_j^0 + \sigma_j^z}{2}$, the projector $\bar{\Pi}_n \equiv \prod\limits_{j=M-n}^{M} \check{q}_j$ cuts out the states with spins down on the last $n+1$ sites of the lattice, $\beta \in \mathbb{C}$, $\widehat{H}$ implies either $\widehat{H}_{XX}$ or $\widehat{H}_{\mathsf{SA}}$, $\boldsymbol{\theta}$ is the solution of the correspondent Bethe equations, and $\mid \Psi_N(\boldsymbol{\theta}) \rangle$ is an eigenstate.

••• XX model

Using the following expressions for the state-vector and its conjugated:

$$|\,\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda}\subseteq\{(M+1-N)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \prod_{k=0}^{M} (\sigma_k^-)^{e_k}\,|\Uparrow\rangle\,,$$

$$\langle\Psi_N(\mathbf{v})\,| = \sum_{\boldsymbol{\lambda}\subseteq\{(M+1-N)^N\}} \langle\Uparrow|\,\prod_{k=0}^{M} (\sigma_k^+)^{\tilde{e}_k}\, S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2})\,.$$

it is easy to calculate the action of the projector:

$$\bar{\Pi}_n\;|\,\Psi_N(\mathbf{u})\rangle \;=\; \sum_{\boldsymbol{\lambda}\subseteq\{(M-N-n)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\left(\prod_{k=0}^{M-n-1} (\sigma_k^-)^{e_k}\right)|\Uparrow\rangle\,.$$

Applying the Binet-Cauchy formula we obtain the following expression for the matrix element:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\Pi}_n \mid \Psi_N(\mathbf{u}) \rangle = \sum_{\boldsymbol{\lambda} \subseteq \{(M-N-n)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}^2)$$

$$= \frac{1}{\mathcal{V}(\mathbf{u}^2)\mathcal{V}(\mathbf{v}^{-2})} \det \left(\frac{1 - (u_k^2/v_j^2)^{(M-n)}}{1 - u_k^2/v_j^2} \right)_{1 \le j,k \le N} .$$

where

$$\mathcal{V}(\mathbf{u}^2) \equiv \prod_{1 \le m < l \le N} (u_l^2 - u_m^2)$$

is the Vandermonde determinant. For $n = -1$ this expression gives the answer for the scalar product of the state-vectors.

The eigenstates of the XX chain form the complete and orthogonal set. It allows to calculate the survival probability:

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) = \frac{e^{\beta E_N^{XX}(\theta)}}{(M+1)^N} \sum_{M \geq k_1 > k_2 \cdots > k_N \geq 0} e^{\beta \sum\limits_{l=1}^{N} \cos(\phi_{k_l})} |\mathcal{V}(e^{i\boldsymbol{\phi}}) \mathcal{P}(e^{-i\boldsymbol{\phi}}, e^{i\boldsymbol{\theta}})|^2 ,$$

where

$$\mathcal{P}(e^{-i\boldsymbol{\phi}}, e^{i\boldsymbol{\theta}}) \equiv \sum_{\boldsymbol{\lambda} \subseteq \{(M-N-n)^N\}} S_{\boldsymbol{\lambda}}(e^{-i\boldsymbol{\phi}})\, S_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\theta}}) ,$$

and $\phi_s = \frac{2\pi}{M+1} s$. We may bring this expression into a more compact form

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) = \frac{e^{\beta E_N^{XX}(\theta)}}{(M+1)^N |\mathcal{V}(e^{i\theta})|^2}$$

$$\times \sum_{M \geq k_1 > k_2 \cdots > k_N \geq 0} e^{\beta \sum\limits_{l=1}^{N} \cos(\phi_{k_l})} \left| \det \left(\frac{1 - (e^{i\theta_p - i\phi_{k_l}})^{M-n}}{1 - e^{i\theta_p - i\phi_{k_l}}} \right) \right|^2_{1 \leq p, l \leq N}$$

We may bring this expression into determinantal form:

$$\boxed{\begin{aligned} \mathcal{T}(\boldsymbol{\theta}, n, \beta) \; &= \; \frac{e^{\beta E_N^{XX}(\theta)}}{(M+1)^N} \times \\ &\times \det_{1 \leq i,j \leq N} \left(\sum_{k,l=0}^{M-n} F_{k;\,l}(\beta)\, e^{-\beta \cos(\theta_i) + i(l\theta_i - k\theta_j)} \right), \end{aligned}}$$

where

$$F_{k;\,l}(\beta) \equiv \frac{1}{M+1} \sum_{s=0}^{M} e^{\beta \cos \phi_s} \, e^{i\phi_s (k-l)}$$

is a generating function of the number of walks with random turns made by a single pedestrian travelling between l^{th} and k^{th} sites of (periodic) chain.

• • • *Izing* limit

The answer for the survival probability is similar to that of the XX model:

$$\boxed{\begin{aligned}\mathcal{T}(\boldsymbol{\theta}, n, \beta) &= \frac{e^{\beta E_N^{SA}(\theta)}}{(M+1)(M+1-N)^{N-1}|\mathcal{V}(e^{i\theta})|^2}\\ &\times \sum_{M-N\geq k_1>k_2\cdots>k_N\geq 0} e^{\beta\sum\limits_{l=1}^{N}\cos(\phi_{k_l})}\left|\det\left(\frac{1-(e^{i\theta_p-i\phi_{k_l}})^{M-N-n+1}}{1-e^{i\theta_p-i\phi_{k_l}}}\right)\right|^2_{1\leq p,l\leq N},\end{aligned}}$$

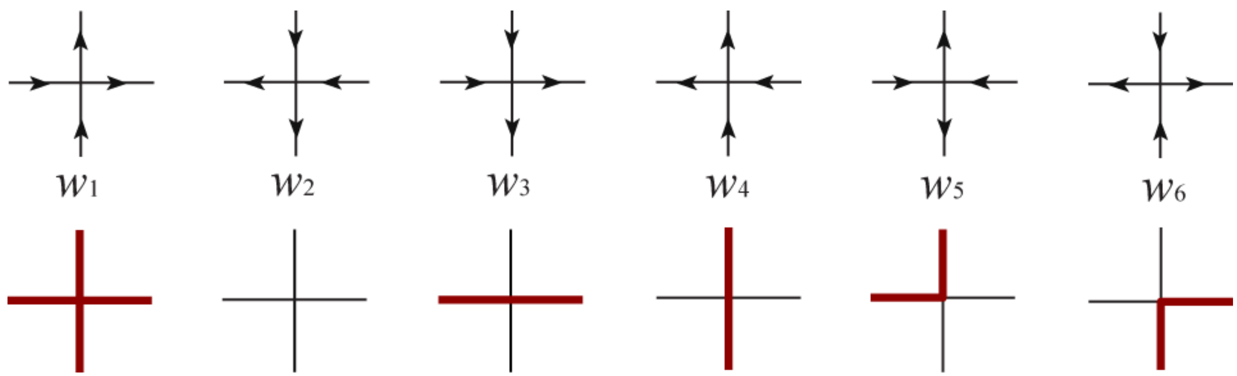
where $\phi_s = \dfrac{2\pi}{M+1-N}\left(s-\dfrac{M-N}{2}\right)$.

When $\beta = 0$ the function $\mathcal{T}(\boldsymbol{\theta}, n, \beta)$ is known as the emptiness formation probability and is equal to

$$\boxed{\begin{aligned}\mathcal{T}(\boldsymbol{\theta}, n) = \frac{M-N+1}{M+1} \times \det\Bigg(\Big(1 - \frac{n}{M-N+1}\Big)\delta_{jk} + \\ + \frac{1-e^{in(\theta_j-\theta_k)}}{(M-N+1)(1-e^{i(\theta_k-\theta_j)})}(1-\delta_{jk})\Bigg)_{1\le k,j\le N}.\end{aligned}}$$

FOUR-VERTEX MODEL AND BOXED PLANE PARTITIONS

A six-vertex model on a square grid is defined by the six different arrows (lines) arrangements. A statistical weight corresponds to each type of the vertices:



The L-operator of the six-vertex model is equal to:

$$L_{6v}(n|u) = \begin{pmatrix} -ue^{\gamma\sigma_n^z} - u^{-1}e^{-\gamma\sigma_n^z} & \sigma_n^- \left(e^{2\gamma} - e^{-2\gamma}\right) \\ \sigma_n^+ \left(e^{2\gamma} - e^{-2\gamma}\right) & ue^{-\gamma\sigma_n^z} + u^{-1}e^{\gamma\sigma_n^z} \end{pmatrix}.$$

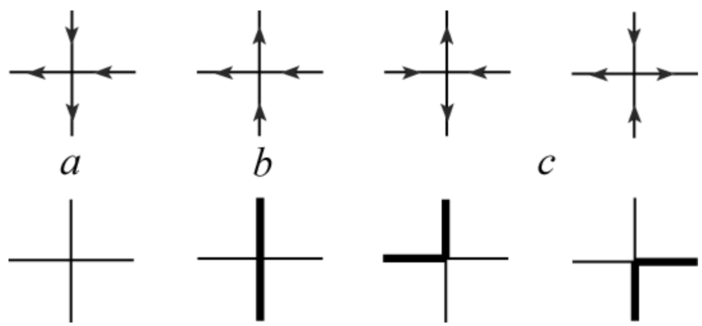
The transfer matrix of the model

$$\tau_{6v}(u) = Tr L_{6v}(M|u) \ldots L_{6v}(1|u) L_{6v}(0|u)$$

commute with Hamiltonian of the XXZ model

$$[\widehat{H}_{XXZ}, \tau_{6v}(u)] = 0, \quad \Delta = -\cosh 2\gamma.$$

The L-operator of the four-vertex model



is obtained as the following limit of the six-vertex L-operator:

$$L_{4v}(n|u) = \lim_{\gamma\to\infty} e^{-2\gamma} e^{\gamma\sigma_n^z} e^{\frac{\gamma}{2}\sigma^z} L_{6v}(n|u) e^{-\frac{\gamma}{2}\sigma^z},$$

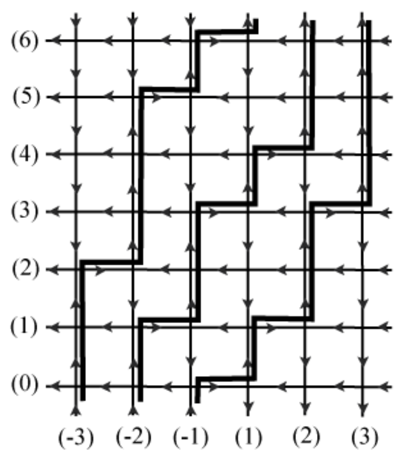
and is equal to

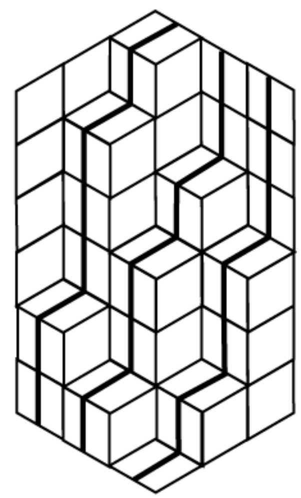
$$L_{4v}(n|u) = \begin{pmatrix} -u\sigma_n^+\sigma_n^- & \sigma_n^- \\ \sigma_n^+ & u^{-1}\sigma_n^+\sigma_n^- \end{pmatrix}.$$

The transfer matrix of the four-vertex model commute with the Hamiltonian in the Izing limit:

$$[\hat{H}_{SA}, \tau_{4v}(u)] = 0.$$

The scalar product the state vectors in the Izing limit may be expressed as the sum over all allowed configurations of vertices on a square lattice with the arrows on first N vertical rows pointing inwards, on the last N ones pointing outwards; on the right and on the left boundaries all arrows are pointing to the left.





Putting $v_j = q^{-\frac{j}{2}}$ and $u_j = q^{\frac{j-1}{2}}$ in the scalar product of the state vectors in the Izing limit we obtain

$$\langle \Psi_N(q^{-\frac{1}{2}}, \ldots, q^{-\frac{N}{2}}) \,|\, \Psi_N(1, \ldots, q^{\frac{N-1}{2}}) \rangle =$$

$$= \sum_{\widetilde{\boldsymbol{\lambda}} \subseteq \{(M+2-2N)^N\}} S_{\widetilde{\boldsymbol{\lambda}}}(q, \ldots, q^N)\, S_{\widetilde{\boldsymbol{\lambda}}}(1, \ldots, q^{N-1}) =$$

$$= q^{-\frac{N}{6}(N-1)(2N-1)} \prod_{1 \le k < j \le N} \left(1 - q^{j-k}\right)^{-2} \det\left(\frac{1 - s^{j+k-1}}{1 - q^{j+k-1}}\right)_{1 \le j,k \le N},$$

where $s = q^{M-N+2}$. The determinant was calculated by G. Kuperberg :

$$\det\left(\frac{1 - s^{j+k-1}}{1 - q^{j+k-1}}\right)_{1 \le j,k \le N} =$$

$$= q^{\frac{N}{6}(N-1)(2N-1)} \prod_{1 \le k < j \le N} \left(1 - q^{j-k}\right)^2 \prod_{k,j=1}^{N} \frac{1 - s q^{j-k}}{1 - q^{j+k-1}},$$

And we obtain $Z_{\mathsf{spp}}(q)$:

$$\boxed{\langle \Psi_N(q^{-\frac{1}{2}}, \ldots, q^{-\frac{N}{2}}) \,|\, \Psi_N(1, \ldots, q^{\frac{N-1}{2}}) \rangle = Z_{\mathsf{spp}}(q)\,,}$$

where $Z_{\mathsf{spp}}(q)$ is the generating functions of the strict plane partitions in a box $N \times N \times M$:

$$Z_{\mathsf{spp}}(q) \;=\; \prod_{1 \le j,k \le N} \frac{1 - q^{M+3-j-k}}{1 - q^{j+k-1}}.$$

For $q = 1$ this formula gives the number of strict plane partitions in a box $N \times N \times M$:

$$Z_{\mathsf{spp}}(1) \;=\; \prod_{1 \le j,k \le N} \frac{M + 3 - j - k}{j + k - 1}.$$

Calculating the scalar product of the state vectors in the XX case with the same parametrization we obtain $Z_{\mathsf{spp}}(q)$:

$$\boxed{\langle \Psi_N(q^{-\frac{1}{2}}, \ldots, q^{-\frac{N}{2}}) \,|\, \Psi_N(1, \ldots, q^{\frac{N-1}{2}}) \rangle = Z_{\mathsf{cspp}}(q)\,,}$$

where $Z_{\mathsf{spp}}(q)$ is the generating functions of the column strict plane partitions in a box $N \times N \times M$:

$$Z_{\mathsf{cspp}}(q) \;=\; \prod_{1 \le j,k \le N} \frac{1 - q^{M+1+j-k}}{1 - q^{j+k-1}}.$$

The number of column strict plane partitions in a box $N \times N \times M$ is equal to

$$Z_{\mathsf{cspp}}(1) \;=\; \prod_{1 \le j,k \le N} \frac{M + 1 + j - k}{j + k - 1}.$$

LOW TEMPERATURE LIMIT

Let us consider the correlation function

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) \equiv \frac{\langle \Psi_N(\boldsymbol{\theta}) \mid \bar{\Pi}_n \, e^{-\beta \hat{H}_{XX}} \, \bar{\Pi}_n \mid \Psi_N(\boldsymbol{\theta}) \rangle}{\langle \Psi_N(\boldsymbol{\theta}) \mid \, e^{-\beta \hat{H}} \, \mid \Psi_N(\boldsymbol{\theta}) \rangle} \, ,$$

over the ground state. In this case $\theta_j = \frac{2\pi}{M+1}\left(N - j - \frac{N-1}{2}\right), \quad 1 \leq j \leq N$. For a long enough chain when $M \gg 1$, while the number N is moderate $1 \ll N \ll M$, we may put $\mathbf{u}^2 \simeq \mathbf{1}$ and obtain

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) = e^{-\beta N} (M+1)^{-N}$$

$$\times \sum_{M \geq k_1 > k_2 \cdots > k_N \geq 0} e^{\beta \sum\limits_{l=1}^{N} \cos(\phi_{k_l})} |\mathcal{V}(e^{i\phi}) \mathcal{P}(e^{-i\phi}, \mathbf{1})|^2 \, ,$$

where

$$\mathcal{P}(e^{-i\boldsymbol{\phi}},\mathbf{1}) \equiv \sum_{\boldsymbol{\lambda}\subseteq\{(M-N-n)^N\}} S_{\boldsymbol{\lambda}}(e^{-i\boldsymbol{\phi}})\, S_{\boldsymbol{\lambda}}(\mathbf{1})\,.$$

In the considered limit we can replace sums by the integrals and obtain:

$$\mathcal{T}(\boldsymbol{\theta},n,\beta)$$

$$\simeq \frac{e^{\beta N}}{N!}\prod_{i=1}^{N}\left(\int_0^{2\pi}\frac{d\phi_i}{2\pi}\right) e^{-\beta\sum\limits_{l=1}^{N}(1-\cos\phi_l)} |\mathcal{P}(e^{-i\boldsymbol{\phi}},\mathbf{1})|^2 \prod_{1\le k<l\le N} |\,e^{i\phi_k}-e^{i\phi_l}\,|^2\ .$$

When β is tending to the infinity this integral can be approximated in the following way:

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) \simeq \mathcal{P}^2(\mathbf{1},\mathbf{1})\,\frac{1}{N!}$$
$$\times\int_0^{2\pi}\cdots\int_0^{2\pi} e^{-(\beta/2)\sum\limits_{l=1}^{N}\phi_l^2} \prod_{1\le k<l\le N} |\phi_k-\phi_l|^2\,\frac{d\phi_1 d\phi_2\ldots d\phi_N}{(2\pi)^N}\,.$$

The integral is the *Mehta integral* of the *Gaussian Unitary Ensemble* of random matrices (M. L. Mehta, *Random Matrices*, Academic Press, London, 1991) and we obtain:

$$\mathcal{T}(\boldsymbol{\theta}, n, \beta) \simeq \mathcal{P}^2(\mathbf{1},\mathbf{1}) \prod_{n=1}^{N} \frac{\Gamma(n)}{(2\pi)^{1/2}} \beta^{-N^2/2}\,.$$

The temperature dependent correlation function calculated over the ground state in the considered limit has the following asymptotical behavior for the XX model

$$\boxed{\mathcal{T}_{XX}(\boldsymbol{\theta}, n, \beta) \simeq \Big(\prod_{k,j=1}^{N} \frac{M-n+j-k}{j+k-1}\Big)^2 \times \prod_{n=1}^{N} \frac{\Gamma(n)}{(2\pi)^{1/2}}\ \beta^{-N^2/2},}$$

and for the Izing limit

$$\boxed{\mathcal{T}_{SA}(\boldsymbol{\theta}, n, \beta) \simeq \Big(\prod_{k,j=1}^{N} \frac{M-N+2-n+j-k}{j+k-1}\Big)^2 \times \prod_{n=1}^{N} \frac{\Gamma(n)}{(2\pi)^{1/2}}\ \beta^{-N^2/2}.}$$

The amplitudes are proportional to the squared numbers of column strict and strict plane partitions in the $N \times N \times (M-N-n)$ box respectively.

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