Deformed Special Relativity (DSR) algebra and its applications

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Outlook

1. Hopf module algebra, smash product, definitions and quasi-deformation
2. various models of κ-Minkowski spacetime
3. Hilbert space realizations and physical applications

κ - Minkowski spacetime

example of **noncommutative spacetimes**.

Two points of view:

Mathematical point of view (Hopf module algebra): twist realization of $\kappa-$Minkowski spacetime described as a quantum covariant algebra, deformation quantization of the corresponding linear Poisson structure.

Physically interesting: κ - Minkowski spacetime provides possible frameworks for deformed (doubly) special relativity theories (DSR).

More concrete..

- **Quantum deformations** which lead to noncommutative spacetimes are strictly connected with quantum groups formalism, which are generalizations of symmetry groups.
- In this way κ-Minkowski spacetime and κ-Poincare algebra are related by the notion of module algebra ($\equiv$ **covariant quantum space**) - algebraic generalization of covariant space.
- Extension κ- Poincaré algebra by κ- Minkowski commutation relations using **crossed (smash) product construction**.
- **Deformation of Weyl subalgebra** provides quantum phase space (crossproduct of κ- Minkowski algebra with algebra of translations) .

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Definitions and examples

Smash products

Hopf algebras

Definitions

Universal enveloping algebra

$\mathcal{U}_{\mathfrak{g}} = \frac{T\mathfrak{g}}{J_{\mathfrak{g}}}$
where $T\mathfrak{g}$-tensor (free) algebra of the vector space $\mathfrak{g}$ and $J_{\mathfrak{g}}$-ideal generated by $< X \otimes Y - Y \otimes X - [X, Y] >$: $X, Y \in \mathfrak{g}$.

Hopf algebra is a structure that is simultaneously a (unital associative) algebra, a (counital coassociative) coalgebra with antipod.

Hopf algebra example

$\mathfrak{g}$ - Lie algebra
$\mathcal{U}_{\mathfrak{g}}$ (its universal enveloping algebra) is Hopf algebra with (for $u \in \mathfrak{g}$):

- the primitive coproduct $\Delta_0(u) = u \otimes 1 + 1 \otimes u$
- counit: $\epsilon(u) = 0, \quad \epsilon(1) = 1$
- antipode: $S_0(u) = -u; \quad S_0(1) = 1$

Hopf module algebra - definition

(Left) **Module algebra** over Hopf algebra $\mathcal{H}$ consist of $\mathcal{H}$-module $\mathcal{A}$ which is simultaneously an unital algebra satisfying the following compatibility condition:
$L \triangleright (f \cdot g) = (L_{(1)} \triangleright f) \cdot (L_{(2)} \triangleright g)$
between multiplication $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$,
coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $\Delta(L) = L_{(1)} \otimes L_{(2)}$,
and (left) module action $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$; for $L \in \mathcal{H}$, $f, g \in \mathcal{A}$,
$L \triangleright 1 = \epsilon(L)$, $1 \triangleright f = f$.

The condition:

$$\boxed{\mathbf{L} \triangleright (\mathbf{f} \cdot \mathbf{g}) = (\mathbf{L}_{(1)} \triangleright \mathbf{f}) \cdot (\mathbf{L}_{(2)} \triangleright \mathbf{g})}$$

generalized Leibniz rule

called **covariance condition** and the corresponding algebra $\mathcal{A}$ a **covariant quantum space** with respect to $\mathcal{H}$.
Primitive elements $\Delta(L) = L \otimes 1 + 1 \otimes L \rightsquigarrow$ normal Leibniz rule.

The covariance condition $(L \triangleright (f \cdot g) = (L_{(1)} \triangleright f) \cdot (L_{(2)} \triangleright g))$ entitles us also to introduce a new unital and associative algebra, the so-called **smash** (or **crossed**) **product** algebra $\mathcal{A} \rtimes \mathcal{H}$.

Smash (crossed) product - definition

Determined on the vector space $\mathcal{A} \otimes \mathcal{H}$ by:
$(f \otimes L)\#(g \otimes M) = f(L_{(1)} \triangleright g) \otimes L_{(2)}M$

Canonical embedding: $\mathcal{A} \ni f \to f \otimes 1$ and $\mathcal{H} \ni L \to 1 \otimes L$ as subalgebras in $\mathcal{A} \rtimes \mathcal{H}$.

Smash $\rightsquigarrow$ tensor product - example

Particularly, the trivial action $L \triangleright g = \epsilon(L)g$ makes $\mathcal{A} \rtimes \mathcal{H}$ isomorphic to the ordinary tensor product algebra $\mathcal{A} \otimes \mathcal{H}$:
$(f \otimes L)\#(g \otimes M) = fg \otimes LM$.

Heisenberg representation - definition

Canonical **Heisenberg representation** on the vector space $\mathcal{A}$ reads as follows:

$$\hat{f}(g) = f \cdot g, \qquad \hat{L}(g) = L \triangleright g$$

where $\hat{f}, \hat{L}$ are linear operators acting in $\mathcal{A}$, i.e. $\hat{f}, \hat{L} \in End\,\mathcal{A}$.

If $\mathcal{A}$ is a **universal envelope** of Lie algebra $\mathfrak{h} \rightsquigarrow \mathcal{A} = \mathcal{U}_{\mathfrak{h}}$.
It is enough to determine the Hopf action on generators $a_i \in \mathfrak{h}$ but with consistency conditions
$\left(L_{(1)} \triangleright a_i\right)\left(L_{(2)} \triangleright a_j\right) - \left(L_{(1)} \triangleright a_j\right)\left(L_{(2)} \triangleright a_i\right) - c_{ij}^k L \triangleright a_k = 0$ hold,
where $[a_i, a_j] = c_{ij}^k a_k$.

Weyl algebra - example of smash product (Heisenberg double)

Weyl algebra as a crossed product of:

an algebra of translations $\mathfrak{T}^n$ containing P_μ generators

an algebra $\mathfrak{X}^n$ of spacetime coordinates x^μ

More strictly, both algebras are defined as a dual pair of the universal commutative algebras with $n-$generators (polynomial algebras), i.e. :

$\mathfrak{T}^n \equiv Poly(P_\mu) \equiv \mathbb{C}[P_0, \ldots P_{n-1}]$

$\mathfrak{X}^n \equiv Poly(x^\mu) \equiv \mathbb{C}[x^0, \ldots x^{n-1}]$

Both algebras are isomorphic to $\mathcal{U}_{\mathfrak{t}^n} \cong \mathfrak{T}^n \cong \mathfrak{X}^n$ of $n-$dimensional Abelian Lie algebra $\mathfrak{t}^n$.

Primitive Hopf algebra structure on $\mathfrak{T}^n \implies$ extend the action implemented by duality map (dual pair of Hopf algebras)

$$P_\mu \triangleright x^\nu = -\imath < P_\mu, x^\nu >= -\imath\delta^\nu_\mu; \qquad P_\mu \triangleright 1 = 0$$

to whole algebra $\mathfrak{X}^n$ due to the Leibniz rule,

for example:
$P_\mu \triangleright (x^\nu x^\lambda) = \delta^\nu_\mu x^\lambda + \delta^\lambda_\mu x^\nu$, induced by primitive coproduct
$\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$.

In result one obtains the following standard set of Weyl-Heisenberg commutation relations:

$$[P_\mu, x^\nu]_\# \equiv [P_\mu, x^\nu] = -\imath\delta^\nu_\mu; \qquad [x^\mu, x^\nu] = [P_\mu, P_\nu] = 0;$$

as generating relations for the Weyl algebra $\mathfrak{W}^n \equiv \mathfrak{X}^n \rtimes \mathfrak{T}^n$.
Notice:
In the Heisenberg representation

$$P_\mu \triangleright = -\imath\partial_\mu = -\imath\frac{\partial}{\partial x^\mu}$$

the Weyl algebra becomes an algebra of differential operators with polynomial coefficients in $\mathbb{R}^n$.

In fact, it is also Heisenberg double obtained from dual pair of Hopf algebras.

Remark

Weyl algebra is not an enveloping algebra of any Lie algebra. Therefore, it makes difficult to determine a Hopf algebra structure on it. The standard way to omit this problem relies on introducing the central element C and replacing Weyl commutation relations by the following ones

$$[P_\mu, x^\nu] = -\imath\delta^\nu_\mu C,$$
$$[x^\mu, x^\nu] = [C, x^\nu] = [P_\mu, P_\nu] = [C, P_\nu] = 0.$$

- The relations above determine $(2n+1)-$dimensional **Heisenberg Lie algebra** of rank $n+1$.
- Thus Heisenberg algebra can be defined as an enveloping algebra for above relations.
- It may provide a starting point for Hopf algebraic deformations (not considered further on here).

Hilbert space representation

This Hopf action extends to the full algebra $C^\infty(\mathbb{R}^n) \otimes \mathbb{C}$ of complex valued smooth functions on $\mathbb{R}^n$. Its invariant subspace of compactly supported functions $C_0^\infty(\mathbb{R}^n) \otimes \mathbb{C}$ form a dense domain in the Hilbert space of square-integrable functions: $\mathcal{L}^2(\mathbb{R}^n, dx^n)$. Consequently, the Heisenberg representation extends to **Hilbert space representation** of $\mathfrak{W}^n$ by (unbounded) operators. This corresponds to canonical quantization procedure and in the relativistic case leads to Stückelberg's version of **Relativistic Quantum Mechanics**

Weyl algebra $\equiv$ quantized phase space

Smash $\rightsquigarrow$ semidirect product - example

$\mathfrak{igl}(n) \equiv \mathfrak{igl}(n, \mathbb{C}) = \mathfrak{gl}(n) \ni \mathfrak{t}^n$

$\mathcal{U}_{\mathfrak{igl}(n)} = \mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)}$

(Left) Hopf $\mathfrak{gl}(n)-$action on $\mathfrak{T}^n$ generators is: $L^\mu_\nu \triangleright P_\rho = \imath \delta^\mu_\rho P_\nu$. Resulting algebra is described by standard set of $\mathfrak{igl}(n)$ commutation relations:

$$[L^\mu_\nu, L^\rho_\lambda] = -\imath \delta^\rho_\nu L^\mu_\lambda + \imath \delta^\mu_\lambda L^\rho_\nu; \qquad [L^\mu_\nu, P_\lambda] = \imath \delta^\mu_\lambda P_\nu, \qquad [P_\mu, P_\nu] = 0$$

Next step: **Weyl extension** of $\mathfrak{gl}(n)$ as a double crossed-product construction $\mathfrak{X}^n \rtimes (\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)})$ with:

$$[L^\mu_\nu, x^\lambda] = -\imath \delta^\lambda_\nu x^\mu, \qquad [P_\mu, x^\nu] = -\imath \delta^\nu_\mu, \qquad [x^\mu, x^\nu] = 0$$

Action is classical: $P_\mu \triangleright x^\nu = -\imath \delta^\nu_\mu, \qquad L^\mu_\nu \triangleright x^\rho = -\imath \delta^\rho_\nu x^\mu$

Weyl algebra $\mathfrak{W}^n$ becomes a subalgebra in $\mathfrak{X}^n \rtimes \left(\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)}\right)$.

Weyl algebra (Heisenberg) realization - definition

Algebra homomorphism $\mathfrak{X}^n \rtimes \left(\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)}\right) \to \mathfrak{W}^n$ provided by

$$P_\mu \to P_\mu, \qquad x^\mu \to x^\mu, \qquad L_\mu^\nu \to x^\nu P_\nu$$

gives **Weyl algebra (Heisenberg) realization** of $\mathfrak{X}^n \rtimes \left(\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)}\right)$.

Particularly, the map $L_\mu^\nu \to x^\nu P_\nu$ is a Lie algebra isomorphism.

Heisenberg representation - definition

The Heisenberg realization described above induces Heisenberg representation of $\mathfrak{X}^n \rtimes \left(\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)}\right)$

$$P_\mu \triangleright = -\imath\partial_\mu \equiv -\imath\frac{\partial}{\partial x^\mu}, \qquad x^\mu = x^\mu, \qquad L_\mu^\nu \triangleright = -\imath x^\nu \partial_\mu$$

acting in the vector space $\mathfrak{X}^n$.

Weyl extension of the Poincare Lie algebra

$\mathfrak{X}^n \rtimes (\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{o}(\eta;n)})$ as subalgebra in $\mathfrak{X}^n \rtimes (\mathfrak{T}^n \rtimes \mathcal{U}_{\mathfrak{gl}(n)})$.

$$[M_{\mu\nu}, M_{\rho\lambda}] = \imath\eta_{\mu\rho}M_{\nu\lambda} - \imath\eta_{\nu\rho}M_{\mu\lambda} - \imath\eta_{\mu\lambda}M_{\nu\rho} + \imath\eta_{\nu\lambda}M_{\mu\rho}$$

$$[M_{\mu\nu}, P_\lambda] = \imath\eta_{\mu\lambda}P_\nu - \imath\eta_{\nu\lambda}P_\mu, \qquad [P_\mu, P_\nu] = 0$$

$$[P_\mu, x_\nu] = -\imath\eta_{\mu\nu}, \qquad [x_\mu, x_\nu] = 0$$

$$[M_{\mu\nu}, x_\lambda] = \imath\eta_{\mu\lambda}x_\nu - \imath\eta_{\nu\lambda}x_\mu$$

where

$$M_{\mu\nu} = \eta_{\mu\lambda}L^\lambda_\nu - \eta_{\nu\lambda}L^\lambda_\mu.$$

In the Heisenberg realization: $M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu$,

$$x_\mu = \eta_{\mu\lambda}x^\lambda\ , \qquad \eta_{\mu\nu} = (-, +, +, \dots)$$

Twist

"deformation theory"

$$Hopf - module(\mathcal{A},\mathcal{H}) \xrightarrow{\qquad\qquad} new\ (deformed)\ objects(\mathcal{A}^{\mathcal{F}},\mathcal{H}^{\mathcal{F}})$$

with the same structure

Notation:

- $\mathcal{A}$- Hopf module algebra over $\mathcal{H}$
- $\mathcal{F}$- twisting element
- $(\mathcal{A}^{\mathcal{F}},\mathcal{H}^{\mathcal{F}})$- deformed Hopf module algebra

Twist-definition

The twisting two-tensor $\mathcal{F}$ is an invertible element in $\mathcal{H} \otimes \mathcal{H}$ which fulfills the 2-cocycle and normalization conditions:

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F} \ , \qquad (\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}$$

which guarantee co-associativity of the deformed coproduct $\Delta^{\mathcal{F}}$ and associativity of the corresponding twisted star-product .

- The algebra $\mathcal{A}^{\mathcal{F}}$ is equipped with a **twisted (deformed) star -product** :$x \star y = m \circ \mathcal{F}^{-1} \triangleright (x \otimes y) = (\bar{\mathrm{f}}^{\alpha} \triangleright x) \cdot (\bar{\mathrm{f}}_{\alpha} \triangleright y)$
- Hopf action $\triangleright$ remains unchanged.
- $\mathcal{F}$ is symbolically written in the following form:
  $\mathcal{F} = \mathrm{f}^{\alpha} \otimes \mathrm{f}_{\alpha} \in \mathcal{H} \otimes \mathcal{H}$ and $\mathcal{F}^{-1} = \bar{\mathrm{f}}^{\alpha} \otimes \bar{\mathrm{f}}_{\alpha} \in \mathcal{H} \otimes \mathcal{H}$

Twisted Hopf algebra -definition reminder

Quantized Hopf algebra $\mathcal{H}^{\mathcal{F}}$ has non-deformed algebraic sector (commutators), while coproducts and antipodes are subject of the deformation:
$\Delta^{\mathcal{F}}(\cdot) = \mathcal{F}\Delta(\cdot)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(\cdot) = u\, S(\cdot)\, u^{-1}$ where $u = \mathrm{f}^{\alpha} S(\mathrm{f}_{\alpha})$.

Twisted smash

Smash product $\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}}$ has deformed cross-commutation relations determined by deformed coproduct $\Delta^{\mathcal{F}}$.

Now determined on the vector space $\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}}$ by the same relation but with $\Delta^{\mathcal{F}}(L) = \sum_i L^{(i)}_{(1)^{\mathcal{F}}} \otimes L^{(i)}_{(2)^{\mathcal{F}}}$:
$(f \otimes L)\#(g \otimes M) = f(L_{(1)^{\mathcal{F}}} \triangleright g) \otimes L_{(2)^{\mathcal{F}}} M.$

Proposition 1: $\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}} \approxeq \mathcal{A} \rtimes \mathcal{H}$.

For any Drinfeld twist $\mathcal{F}$ the twisted smash product algebra $\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}}$ is isomorphic to the initial (undeformed) one $\mathcal{A} \rtimes \mathcal{H}$. In other words the algebra $\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}}$ is twist independent and can be realized by a change of generators in the algebra $\mathcal{A} \rtimes \mathcal{H}$.

but subalgebras $\mathcal{A}$ and $\mathcal{A}^{\mathcal{F}}$ are not isomorphic.

sketch of the proof

- $\forall x \in \mathcal{A}$ one can consider $\tilde{x} = (\bar{\mathrm{f}}^{\alpha} \triangleright x) \cdot \bar{\mathrm{f}}_{\alpha} \in \mathcal{A} \rtimes \mathcal{H}$.
- Then $\tilde{x} \cdot \tilde{y} = (\bar{\mathrm{f}}^{\alpha} \triangleright (x \star y)) \cdot \bar{\mathrm{f}}_{\alpha}$ and subalgebra generated by elements $\tilde{x}$ is isomorphic to $\mathcal{A}^{\mathcal{F}}$.
- isomorphism is defined on generators by
  $\mathcal{A}^{\mathcal{F}} \ni x \to \tilde{x} \in \mathcal{A} \rtimes \mathcal{H}$ and $\mathcal{H}^{\mathcal{F}} \ni L \to L \in \mathcal{A} \rtimes \mathcal{H}$
- due to invertibility of twist: $x = (f^{\alpha} \triangleright \tilde{x}) \star f_{\alpha}$

and then the isomorphism can be described as a change of generators ("basis"): $(x^{\mu}, L^{k}) \to (\hat{x}^{\mu}, L^{k})$ in $\mathcal{A} \rtimes \mathcal{H}$.

- Twisted deformation of some Lie algebra $\mathfrak{g}$ requires a topological extension of the corresponding enveloping algebra $\mathcal{U}_{\mathfrak{g}}$ into an **algebra of formal power series** $\mathcal{U}_{\mathfrak{g}}[[h]]$ in the formal parameter h (twisting element has to be invertible).
- Hopf module algebra $\mathcal{A}$ has to be extended by $h-$adic topology to $\mathcal{A}[[h]]$ as well.

h-adic topology primer

Elements of the $\mathbb{C}[[h]]$ are in the form:

$$\mathbb{C}[[h]] \ni a = \sum_{n=0}^{\infty} a_n h^n$$

where a_n are complex coefficients and h is indeterminate.
One can also see this ring as $\mathbb{C}[[h]] = \times_{n=0}^{\infty}\mathbb{C}$ which elements are (infinite) sequences of complex numbers $(a_0, a_1, ..., a_n, ...)$ with powers of h just "enumerating" the position of the coefficient.

The structure of ring $\mathbb{C}[[h]]$ is determined by:
$a + b := \sum_{n=0}^{\infty}(a_n + b_n)h^n, \qquad a \cdot b := \sum_{n=0}^{\infty}(\sum_{r+s=n}^{\infty} a_r b_s)h^n.$
The ring is equipped with the so-called **"h-adic" topology**:

- which is determined a by h-adic **"ultra-norm"** $||\cdot||_{ad}$ which is defined by:
$$||\sum_{n=0}^{\infty} a_n h^n||_{ad} = 2^{-n(a)}$$
where $n(a)$ is the smallest integer such that $a_n \neq 0$ (for $a \equiv 0$ one sets $n(a) = \infty$ and therefore $||0||_{ad} = 0$).
- with the properties:
$0 \leq ||a||_{ad} \leq 1 \qquad ||a+b||_{ad} \leq max(||a||_{ad}, ||b||_{ad})$
$||a \cdot b||_{ad} = ||a||_{ad}||b||_{ad} \qquad ||h^k||_{ad} = 2^{-k}$
- This norm is discrete (with values in inverse powers of 2).
- The elements $a \in \mathbb{C}[[h]]$ are invertible if an only if $||a||_{ad} = 1$.
- The ring $\mathbb{C}[[h]]$ is **complete in h-adic topology**.

Topologically free module-definition

Considering V as complex vector space the set $V[[h]]$ contains all formal power series $v = \sum_{n=0}^{\infty} v_n h^n$ with coefficients $v_n \in V$. Therefore $V[[h]]$ is a $\mathbb{C}[[h]]$-module. More generally in the deformation theory we are forced to work with the category of $\mathbb{C}[[h]]$-modules. $V[[h]]$ provides an example of **topologically free modules**. Particularly if V is finite dimensional it is also free module. Any basis $(e_1, ..., e_N)$ in V serves as a system of free generators in $V[[h]]$. More exactly

$$\sum_{k=0}^{\infty} v_k h^k = \sum_{a=1}^{N} x^a e_a$$

where the coordinates $x^a = \sum_{n=0}^{\infty} x_n^a h^n \in \mathbb{C}[[h]]$. It shows that $V[[h]]$ is canonically isomorphic to $V \otimes \mathbb{C}[[h]]$.

Twist deformation of Weyl algebra-example

h-adic extension $\mathcal{U}_{\mathfrak{igl}(n)} \to \mathcal{U}_{\mathfrak{igl}(n)}[[h]] \Longrightarrow \mathfrak{X}^n \to \mathfrak{X}^n[[h]]$, which remains to be (undeformed) module algebra under $\mathbb{C}[[h]]$-extended Hopf action $\triangleright$.

Smash product contains h-adic extension of the Weyl algebra:

$$\mathfrak{W}^n[[h]] = \mathfrak{X}^n[[h]] \rtimes \mathfrak{T}^n[[h]] \subset \mathfrak{X}^n[[h]] \rtimes \left(\mathfrak{T}^n[[h]] \rtimes \mathcal{U}_{\mathfrak{gl}(n)}[[h]]\right)$$

$\mathfrak{igl}(n)$−twist $\mathcal{F}$ can deform simultaneously both:
$U_{\mathfrak{igl}(n)}[[h]] \mapsto U_{\mathfrak{igl}(n)}[[h]]^{\mathcal{F}}$ and $\mathfrak{X}^n[[h]] \mapsto (\mathfrak{X}^n[[h]])^{\mathcal{F}}$ keeping the Hopf action $\triangleright$ unchanged.

Deformed algebra $\mathfrak{X}^n[[h]]^{\mathcal{F}}$ has deformed star multipication $\star$ and can be represented by deformed $\star-$commutation relations

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \imath\, h\theta^{\mu\nu}(x) \equiv \imath\, h(\theta^{\mu\nu} + \theta^{\mu\nu}_\lambda x^\lambda + \dots)$$

replacing the undeformed (commutative) ones where the coordinate functions (x^μ) play a role of generators for the corresponding algebras: deformed and undeformed one.
Deformed Weyl algebra:

$$\mathfrak{W}^n[[h]]^{\mathcal{F}} = \mathfrak{X}^n[[h]]^{\mathcal{F}} \rtimes \mathfrak{T}^n[[h]]^{\mathcal{F}}$$

$\mathfrak{T}^n[[h]]^{\mathcal{F}}$ denotes the corresponding Hopf subalgebra of deformed momenta in $\left(\mathfrak{T}^n[[h]] \rtimes \mathcal{U}_{\mathfrak{gl}(n)}[[h]]\right)^{\mathcal{F}}$.

Proposition 2: $\mathfrak{X}^n[[h]] \rtimes \mathcal{U}_{\mathfrak{igl}(n)}[[h]] \approxeq \mathfrak{X}^n[[h]]^{\mathcal{F}} \rtimes (\mathcal{U}_{\mathfrak{igl}(n)}[[h]])^{\mathcal{F}}$

Proposition 1 ($\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}} \approxeq \mathcal{A} \rtimes \mathcal{H}$)implies that $\mathfrak{X}^n[[h]] \rtimes \mathcal{U}_{\mathfrak{igl}(n)}[[h]]$ is $\mathbb{C}[[h]]$ isomorphic to $\mathfrak{X}^n[[h]]^{\mathcal{F}} \rtimes (\mathcal{U}_{\mathfrak{igl}(n)}[[h]])^{\mathcal{F}}$. Moreover, this isomorphism is congruent to the identity map modulo h.

Proposition 3: $\mathfrak{W}^n[[h]]^{\mathcal{F}} \approxeq \mathfrak{W}^n[[h]]$

All $\mathfrak{igl}(n)-$twist deformed Weyl algebras $\mathfrak{W}^n[[h]]^{\mathcal{F}}$ are $\mathbb{C}[[h]]$-isomorphic to undeformed h-adic extended Weyl algebra $\mathfrak{W}^n[[h]]$.

Pseudo-deformation -definition

In this sense we can say that $\mathfrak{W}^n[[h]]^{\mathcal{F}}$ is **pseudo-deformation** of $\mathfrak{W}^n[[h]]$ since the latter one can be obtained by (nonlinear and invertible) change of generators from the first one.

Remark

The deformed algebra $\mathfrak{X}^n[[h]]^{\mathcal{F}}$ makes up a **deformation quantization** of $\mathfrak{X}^n$ equipped with the Poisson structure (brackets)

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu}(x) \equiv \theta^{\mu\nu} + \theta^{\mu\nu}_\lambda x^\lambda + \theta^{\mu\nu}_{\lambda\rho} x^\lambda x^\rho \dots$$

represented by Poisson bivector $\theta = \theta^{\mu\nu}(x)\partial_\mu \wedge \partial_\nu$. We assume that $\theta^{\mu\nu}(x)$ are polynomial functions, where $\theta^{\mu\nu}, \theta^{\mu\nu}_\lambda, \theta^{\mu\nu}_{\lambda\rho} \dots$ are real numbers.

$\kappa-$Minkowski Poisson structure

Covariant deformation quantization of the $\kappa-$Minkowski Poisson structure is represented by the linear Poisson bivector
$\theta_{\kappa M} = x^k \partial_k \wedge \partial_0 \ , \qquad k = 1 \dots n-1$ on $\mathfrak{X}^n$.
There is one-to-one correspondence between linear Poisson structure $\theta = \theta_\lambda^{\mu\nu} x^\lambda \partial_\mu \wedge \partial_\nu$ on $\mathfrak{X}^n$ and $n-$dimensional Lie algebra $\mathfrak{g} \equiv \mathfrak{g}(\theta)$

$$[X^\mu, X^\nu] = \imath\, \theta_\lambda^{\mu\nu} X^\lambda$$

with the constants $\theta_\lambda^{\mu\nu}$ playing the role of structure constants. Corresponding universal enveloping algebra $\mathcal{U}_\mathfrak{g}$ is **canonical** quantization of $(\mathfrak{g}^*, \theta)$ (Kontsevich).

- One can consider the following modification of the universal enveloping algebra construction $\mathcal{U}_h(\mathfrak{g}) = \frac{T\mathfrak{g}[[h]]}{J_h}$ where J_h denotes an ideal generated by elements $< X \otimes Y - Y \otimes X - h[X, Y] >$ and is closed in h-adic topology
- In other words the algebra $\mathcal{U}_h(\mathfrak{g})$ is h-adic unital algebra generated by $h-$shifted relations $[\mathcal{X}^\mu, \mathcal{X}^\nu] = \imath\, h\theta^{\mu\nu}_\lambda \mathcal{X}^\lambda$ imitating the Lie algebraic ones ($[X^\mu, X^\nu] = \imath\, \theta^{\mu\nu}_\lambda X^\lambda$).
- This algebra provides the so-called **universal** quantization of $(\mathfrak{X}^n, \theta)$
- Moreover, There is a surjective algebra homomorphism $\mathcal{U}_h(\mathfrak{g}) \longrightarrow (\mathfrak{X}^n[[h]])^{\mathcal{F}}$ for a suitable twist $\mathcal{F}$
- In the the case of $\theta_{\kappa M}$ the corresponding Lie algebra is a solvable one
- We denote it as $\mathfrak{an}^{n-1}$ (Borel subalgebra of $\mathfrak{o}(1, n)$) (dim n, rank $n-1$
- its Lie algebraic counterpart the canonical $\kappa-$ Minkowski spacetime algebra $\mathcal{U}_{\mathfrak{an}^{n-1}}$ (will be introduced later on).

Kappa- Minkowski spacetime

1 Twisted

2 universal

3 Canonical

Twisted Kappa- Minkowski spacetime - as $\mathfrak{igl}(n)$ Hopf module algebra

Two one-parameter families of twists corresponding to twisted star-product realization of the $\kappa-$deformed Minkowski spacetime algebra.
$(\mathfrak{X}^n[[h]])^{\mathcal{F}}$ is generated by relations:

$$[x^0, x^k]_\star = \imath\, h x^k; \qquad [x^k, x^j]_\star = 0\ , \qquad k, j = 1 \dots n-1$$

and constitutes a covariant algebra over deformed $\mathfrak{igl}(n)[[h]]$.
Notice:

1. h is the formal parameter. So above algebra **is not** a Lie algebra
2. the result of star multiplication of two generators $x^\mu \star x^\nu$ is explicitly twist dependent, but the generating relations (commutators) are twist independent
3. all algebras $(\mathfrak{X}^n[[h]])^{\mathcal{F}}$ are mutually isomorphic to each other

Abelian family of twists providing $\kappa-$Minkowski spacetime

$$\mathfrak{A}_s = \exp\left[\imath\, h\left(sP_0 \otimes D - (1-s)\, D \otimes P_0\right)\right]$$

Note:

- All twists correspond to the same classical r-matrix: $\mathfrak{r} = D \wedge P_0$
- and they have the same universal quantum r-matrix which is of exponential form: $\mathcal{R} = \mathfrak{A}_s^{21}\mathfrak{A}_s^{-1} = e^{\imath D \wedge P_0}$
- The Heisenberg realization of it coincides with the Poisson bivector.
- This implies that corresponding Hopf algebra deformations of $\mathcal{U}_{\mathfrak{igl}(n)}[[h]]^{\mathcal{F}}$ for different values of parameter s are isomorphic.
- Indeed, $\mathfrak{A}_s$ are related by trivial twist: $\mathfrak{A}_{s_2} = \mathfrak{A}_{s_1}\mathcal{F}_{W_{12}}$, where $W_{12} = \exp\left(\imath(s_1 - s_2)aDP_0\right)$.

s-deformed phase space -Abelian twist

Smash product construction (for given Δ_s) together with classical action leads to crossed commutators:

$$[\hat{x}^\mu, P_0]_s = i\delta_0^\mu; \qquad [\hat{x}^\mu, P_k]_s = i\delta_k^\mu e^{h(1-s)P_0} - ihs\delta_0^\mu P_k$$

Together with commutators involving $\hat{x}^\mu, L_\mu^\nu$, κ- Minkowski spacetime and $\mathfrak{igl}(n)$ relations they form: $\mathfrak{X}^n[[h]]^{\mathcal{F}} \rtimes \mathcal{U}_{\mathfrak{igl}(n)}[[h]]^{\mathcal{F}}$. The change of generators $(L_\mu^\nu, P_\rho, x^\lambda) \to (L_\mu^\nu, P_\rho, \hat{x}_s^\lambda)$, where

$$\hat{x}_s^i = x^i e^{(1-s)hP_0}, \qquad \hat{x}_s^0 = x^0 - hs\, D$$

implies the isomorphism from Proposition 1 ($\mathcal{A}^{\mathcal{F}} \rtimes \mathcal{H}^{\mathcal{F}} \approxeq \mathcal{A} \rtimes \mathcal{H}$). Heisenberg representation acting in the vector space $\mathfrak{X}^n[[h]]$:

$$\hat{x}_s^i = x^i e^{(1-s)hP_0}, \qquad \hat{x}_s^0 = x^0 - hsx^k P_k$$

give rise to Hilbert space extension acting in $\mathcal{L}^2(\mathbb{R}^n, dx^n)[[h]]$ provided that $P_k = -\imath\partial_k$.

Jordanian family for $\kappa-$Minkowski spacetime

Jordanian twists have the following form:

$$\mathfrak{J}_r = \exp\left(J_r \otimes \sigma_r\right)$$

where$J_r = \imath(\frac{1}{r}D - L_0^0)$ with a numerical factor $r \neq 0$ labeling different twists and $\sigma_r = \ln(1 - h\,rP_0)$.

- The corresponding classical r-matrices are the following: $\mathfrak{r}_J = \mathfrak{J}_r \wedge P_0 = \frac{1}{r}D \wedge P_0 - L_0^0 \wedge P_0$
- For different values of the paprameter r classical $r-$matrices are not the same.

$r-$deformed phase space $\mathfrak{W}^n[[h]]^{\mathcal{F}}$

Smash product construction (for fixed value of the parameter r) of momentum algebra with κ- Minkowski algebra gives the following crossed commutators:

$$[\hat{x}^\mu, P_0]_r = i\delta_0^\mu e^{\sigma_r} = i\delta_0^\mu(1 - hrP_0); \quad [\hat{x}^\mu, P_k]_r = i\delta_k^\mu(1 - hrP_0)^{-\frac{1}{r}}$$

Jordanian Heisenberg realizations

Position-momentum-$\mathfrak{gl}(n)$ algebra extension one gets adding to above $r-$deformed phase space commutators containing $\hat{x}^{\mu}$, L^{ν}_{μ}. Heisenberg realization is now in the following form:

$\hat{x}^i_r = x^i\,(1 - raP_0)^{-\frac{1}{r}}$, and $\hat{x}^0_r = x^0(1 - raP_0)$.

Note:

- the above formulas take the same form before and after Heisenberg realization
- that commutation relation ($[\hat{x}^{\mu}, P_0]_r = i\delta^{\mu}_0 e^{\sigma_r} = i\delta^{\mu}_0(1 - hrP_0)$; $[\hat{x}^{\mu}, P_k]_r = i\delta^{\mu}_k(1 - hrP_0)^{-\frac{1}{r}}$) can be reached by this nonlinear change of generators: $(P_\rho, x^\lambda) \to (P_\rho, \hat{x}^\lambda_r)$.
- This illustrates Propositions 2 $(\mathfrak{X}^n[[h]] \rtimes \mathcal{U}_{\mathfrak{igl}(n)}[[h]] \approxeq \mathfrak{X}^n[[h]]^{\mathcal{F}} \rtimes (\mathcal{U}_{\mathfrak{igl}(n)}[[h]])^{\mathcal{F}})$ and 3$(\mathfrak{W}^n[[h]]^{\mathcal{F}} \approxeq \mathfrak{W}^n[[h]])$ for the Jordanian case.
- In the Heisenberg realization the classical $r-$matrices coincide with Poisson bivector

2 Universal Kappa- Minkowski spacetime

as κ- Poincaré Hopf module algebra

h-adic κ- Poincaré Hopf module algebra

Intro:

1. Quantum deformations of Lie algebra are controlled by classical r -matrices
2. r -matrices satisfying classical homogeneous Yang-Baxter (YB) equation the co-algebraic sector is twist-deformed while algebraic one remains classical and twisting Hopf modules (previous case: Twisted Kappa- Minkowski)
3. For inhomogeneous r -matrices one applies Drinfeld-Jimbo quantization instead:

   simultaneous deformations of the algebraic and coalgebraic sectors and applies to semisimple Lie algebras
4. it implies existence of classical basis for Drinfeld-Jimbo quantized algebras

1. Drinfeld-Jimbo procedure cannot be applied to the Poincaré non-semisimple algebra
2. Nevertheless, quantum $\kappa-$Poincaré group shares many properties of the original Drinfeld-Jimbo quantization
3. These include existence of classical basis, the square of antipode and solution to specialization problem
4. There is no cocycle twist related with Drinfeld-Jimbo deformation. Drinfeld-Jimbo quantization has many non-isomorphic forms

$\kappa-$Poincaré with "h-adic" topology

$\mathfrak{io}(1,3)$ with a convenient choice of "physical" generators (M_i, N_i, P_μ):

$$[M_i,\ M_j] = \imath\,\epsilon_{ijk} M_k;\ \ [M_i,\ N_j] = \imath\epsilon_{ijk}\,N_k;$$
$$[N_i, N_j] = -\imath\epsilon_{ijk} M_k;\ \ [P_\mu, P_\nu] = [M_j, P_0] = 0;$$
$$[M_j, P_k] = \imath\epsilon_{jkl} P_l;$$
$$[N_j,\ P_k] = -\imath\delta_{jk}\,P_0, \quad [N_j, P_0] = -\imath\,P_j$$

Hopf algebra can be defined on $\mathcal{U}_{\mathfrak{io}(1,3)}[[h]]$ by establishing deformed coproducts of generators:

$\Delta_\kappa\left(M_i\right) = \Delta_0\left(M_i\right) = M_i \otimes 1 + 1 \otimes M_i$

$\Delta_\kappa\left(N_i\right) =$

$N_i \otimes 1 + \left(hP_0 + \sqrt{1 - h^2 P^2}\right)^{-1} \otimes N_i - h\epsilon_{ijm} P_j \left(hP_0 + \sqrt{1 - h^2 P^2}\right)^{-1} \otimes M_m$

$\Delta_\kappa\left(P_i\right) = P_i \otimes \left(hP_0 + \sqrt{1 - h^2 P^2}\right) + 1 \otimes P_i$

$\Delta_\kappa\left(P_0\right) = P_0 \otimes \left(hP_0 + \sqrt{1 - h^2 P^2}\right) + \left(hP_0 + \sqrt{1 - h^2 P^2}\right)^{-1} \otimes P_0 + hP_m \left(hP_0 + \sqrt{1 - h^2 P^2}\right)^{-1} \otimes P^m$

and deformed antipodes.

- **Note: above expressions are formal power series in the formal parameter h.**
- This determines celebrated $\kappa-$Poincaré quantum group on $\mathcal{U}_{\mathfrak{io}(1,3)}[[h]]$
  with parameter h of $[lenght] = [mass]^{-1}$ dimension. In contrast to the original form the coproduct is written in the classical Poincaré basis.
- Since it is Drinfeld-Jimbo type deformation with h-adic topology we shall denote it as $\mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$.

Since κ- Poincaré Hopf algebra presented above is not obtained by twist deformation one needs new construction of κ-Minkowski spacetime as a $U_{\mathfrak{io}(1,3)}[[h]]^{DJ}$ (Hopf) module algebra.

"h-adic" universal $\kappa-$Minkowski spacetime

$\mathcal{U}_h(\mathfrak{an}^3)$-universal $\kappa-$Minkowski spacetime
with defining relations:

$$[\mathcal{X}^0, \mathcal{X}^i] = \imath\, h\, \mathcal{X}^i; \qquad [\mathcal{X}^j, \mathcal{X}^k] = 0$$

Note: Due to universal construction there is a $\mathbb{C}[[h]]-$algebra epimorphism of $\mathcal{U}_h(\mathfrak{an}^3)$ onto $\mathfrak{X}^4[[h]]^{\mathcal{F}}$ for any $\kappa-$Minkowski twist $\mathcal{F}$. The algebra $\mathfrak{X}^4[[h]]$ is by construction a topologically free $\mathbb{C}[[h]]-$module. For the case of $\mathcal{U}_h(\mathfrak{an}^3)$ this question is open.

Before using smash product construction one has to assure that $\kappa-$Minkowski algebra $\mathcal{U}_h(\mathfrak{an}^3)$ is $\mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$-module.

Remark

As it was already noticed that the algebra $\mathcal{U}_h(\mathfrak{an}^3)$ is different than $\mathcal{U}_{\mathfrak{an}^3}[[h]]$. Assuming $P_\mu \triangleright \mathcal{X}^\nu = \imath a \delta^\nu_\mu$ and $[\mathcal{X}^0, \mathcal{X}^k] = \imath b \mathcal{X}^k$ one gets the following relation:

$$b = -ah$$

Particulary, our choice $a = -1$ does imply $b = h$.
In contrast $b = 1$ requires $a = h^{-1}$ what is not possible for formal parameter h. This explains why the classical action cannot be extended to the unshifted generators X^ν and entire algebra $\mathcal{U}_{\mathfrak{an}^3}[[h]]$. The last one seemed to be the most natural candidate for $\kappa-$Minkowski spacetime algebra in the h-adic case.

DSR algebra

DSR algebra as a crossed product extension of κ -Minkowski and Poincaré algebras: $\mathcal{U}_h(\mathfrak{an}^3) \rtimes \mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$ with cross-commutation relations:

$[M_i, \mathcal{X}_0] = 0 \quad [N_i, \mathcal{X}_0] = -\imath\mathcal{X}_i - \imath h N_i, [M_i, \mathcal{X}_j] = \imath\epsilon_{ijk}\mathcal{X}_k$

$[N_i, \mathcal{X}_j] = -\imath\delta_{ij}\mathcal{X}_0 + \imath h\epsilon_{ijk}M_k$

$[P_k, \mathcal{X}_0] = 0, \qquad [P_k, \mathcal{X}_j] = -\imath\delta_{jk}\left(hP_0 + \sqrt{1-h^2P^2}\right)$

$[P_0, \mathcal{X}_j] = -\imath hP_j, \qquad [P_0, \mathcal{X}_0] = \imath\sqrt{1-h^2P^2}$

in a covariant form:

$$[M_{\mu\nu}, \mathcal{X}_\lambda] = \imath\,\eta_{\mu\lambda}\mathcal{X}_\mu - \imath\,\eta_{\nu\lambda}\mathcal{X}_\mu - \imath\,a_\mu M_{\nu\lambda} + \imath\,a_\nu M_{\mu\lambda}$$

where $a_\mu = \eta_{\mu\nu}a^\nu; (a^\nu) = (h, 0, 0, 0)$.

$\mathcal{U}_h(\mathfrak{an}^3) \rtimes \mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$ as pseudo-deformation of Weyl-extended Poincare

DSR algebra $\mathcal{U}_h(\mathfrak{an}^3) \rtimes \mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$ as introduced above is a deformation of the inhomogeneous special orthogonal algebra for the Lorentzian, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$, case.
The latter can be obtained as a limit of the former when $h \to 0$ by nonlinear change of generators in undeformed algebra.
To this aim we use covariant Heisenberg realizations proposed by Meljanac et al.: $\hat{\mathcal{X}}^\mu = x^\mu \left(hp_0 + \sqrt{1 - h^2 p^2} \right) - hx^0 p^\mu$;
$\hat{M}_{\mu\nu} = M_{\mu\nu}; \qquad \hat{P}_\mu = p_\mu$.
But we do not require Heisenberg realization for $M_{\mu\nu}$ (In Heisenberg realization $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$).
There exist huge class of Heisenberg realizations of above DSR algebra $\mathcal{U}_h(\mathfrak{an}^3) \rtimes \mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$.

Why one considers deformed Casimir for $\kappa-$Poincaré?

- The center of the algebra $\mathcal{U}_{\mathfrak{io}(1,3)}[[h]]^{DJ}$ is an algebra over $\mathbb{C}[[h]]$.
- Therefore one can consider deformation of the classical Poincaré Casimir operator P^2. In fact for any power series of two variables $f(s,t)$ the element: $\mathcal{C}_f = f(P^2, h) \in \mathscr{Z}$. The reason of using deformed Casimir is that the standard one fails to satisfy
$$[P^2, \mathcal{X}^\mu] = 2P^\mu$$
due to noncommutativity of $\mathcal{X}_\mu$.
- Considering deformed Casimir one has freedom to choose the form of function f. The choice
$$\mathcal{C}_h^2 = 2h^{-2}\left(\sqrt{1 + h^2P^2} - 1\right)$$

preserves the classical properties:
$$[M_{\mu\nu}, \mathcal{C}_h^2] = [\mathcal{C}_h^2, P_\mu] = 0; \quad [\mathcal{C}_h^2, \mathcal{X}_\mu] = 2P_\mu$$

Dispersion relations

The standard Poincaré Casimir gives rise to undeformed dispersion relation:

$$P^2 + m_{ph}^2 = 0$$

where m_{ph} is mass parameter. The second Casimir operator leads to deformed dispersion relations

$$\mathcal{C}_h^2 + m_h^2 = 0$$

with the deformed mass parameter m_h. Relation between this two mass parameters has the following form:

$$m_{ph}^2 = m_h^2(1 - \frac{h^2}{4} m_h^2)$$

For photons $m_{ph} = m_h = 0$.

Realizations

Particularly important is the so-called **non-covariant family of realizations**, labeled by two arbitrary (analytic) functions ψ, γ.

Convenient notation

For a given (analytic) function $f(t) = \sum f_n t^n$ of one variable we shall denote by

$$\tilde{f} = f(-hp_0) = \sum f_n(-1)^n p_0^n h^n \in \mathfrak{W}^4 \tag{1}$$

the corresponding element $\tilde{f} \in \mathfrak{W}^4[[h]]$.
We will use also:

$$\Psi(t) = \exp\left(\int_0^t \frac{dt'}{\psi(t')}\right); \quad \Gamma(t) = \exp\left(\int_0^t \frac{\gamma(t')dt'}{\psi(t')}\right)$$

for an arbitrary choice of ψ, γ such that $\psi(0) = 1$.

Generators of deformed Weyl algebra $\mathcal{U}_h(\mathfrak{an}^3) \rtimes \mathfrak{T}[[h]]^{DJ}$

admit the following Heisenberg realization:

$$\mathcal{X}^i = x^i \tilde{\Gamma} \tilde{\Psi}^{-1}, \quad \mathcal{X}^0 = x^0 \tilde{\psi} - h x^k p_k \tilde{\gamma}$$

together with

$$P_i = p_i \tilde{\Gamma}^{-1}, \qquad P_0 = h^{-1} \tfrac{\tilde{\Psi}^{-1} - \tilde{\Psi}}{2} + \tfrac{1}{2} h \vec{p}^2 \tilde{\Psi} \tilde{\Gamma}^{-2}$$

. The remaining DSR algebra generators in terms of undeformed Weyl algebra $\mathfrak{W}^4[[h]]$-generators (x^μ, p_ν) are the following:

$$M_i = \imath \epsilon_{ijk} x_j p_k = \imath \epsilon_{ijk} \mathcal{X}_j P_k \tilde{\Psi}$$

$$N_i = h^{-1} x_i \tilde{\Gamma} \frac{\tilde{\Psi}^{-1} - \tilde{\Psi}}{2} - x_0 p_i \tilde{\psi} \tilde{\Psi} \tilde{\Gamma}^{-1} + \frac{\imath}{2} h x_i \vec{p}^2 \tilde{\Psi} \tilde{\Gamma}^{-1} - h x^k p_k p_i \tilde{\gamma} \tilde{\Psi} \tilde{\Gamma}^{-1}$$
$$= (\mathcal{X}_i P_0 - \mathcal{X}_0 P_i) \tilde{\Psi}$$

The deformed Casimir operator reads as:

$$\mathcal{C}_h = h^{-2}(\tilde{\Psi}^{-1} + \tilde{\Psi} - 2) - \vec{p}^2\tilde{\Psi}\tilde{\Gamma}^{-2}$$

Above realization has proper classical limit:
$\mathcal{X}^\mu = x^\mu, \quad M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad P_\mu = p_\mu$ as $h \to 0$. Moreover measurable frame is provided by (commuting) position x^μ operators and local, physically measurable, momentum $p_\mu = -i\partial_\mu$, which is important in DSR theories interpretation.

Commutative coordinates

besides $\kappa-$Minkowski coordinates $\mathcal{X}^\mu$ one can also introduce usual (commuting) Minkowski like coordinates $\tilde{x}^\mu \doteq \mathcal{X}^\mu \Psi^{-1}$ which differ from x^μ. The rotation and boost generators expressed above take then the familiar form:

$$M_i = \imath\epsilon_{ijk}\tilde{x}_j P_k \qquad \text{and} \qquad N_i = (\tilde{x}_i P_0 - \tilde{x}_0 P_i)$$

All twisted realizations are special case of the one above for special choice of the functions ψ, γ.

Abelian realization

$$\hat{x}^i_s = x^i e^{(1-s)hP_0}, \qquad \hat{x}^0_s = x^0 - hsx^k P_k$$

one gets taking constant functions $\psi = 1$ and $\gamma = s$. Hermiticity of $\hat{x}^0$ forces $\gamma = 0$.

Jordanian realization

$$\hat{x}^i_r = x^i \left(1 - raP_0\right)^{-\frac{1}{r}}, \qquad \hat{x}^0_r = x^0(1 - raP_0)$$

requires $\psi = 1 + rt$ and $\gamma = 0$.

Physical consequences of DSR algebra formalism

d'Alembert operator is played by Casimir operator of κ Poincare algebra:

$$\left(\mathcal{C}_\kappa - m_\kappa^2\right)\omega_p = 0$$

where $\omega_p = \exp\left(\imath\, p_\mu x^\mu\right)$ represents the plane wave with the wave vector $p = (p_\mu)$.
For photons: $m = m_\kappa = 0$ deformed Klein-Gordon equation puts constraint on wave vector p_μ in the following form of dispersion relation:

$$|\vec{p}| = -\kappa\left(1 - \exp\left(-\int_0^{-\frac{p_0}{\kappa}} \frac{da}{\psi(a)}\right)\right)\exp\left(\int_0^{-\frac{p_0}{\kappa}} \frac{\gamma(a)da}{\psi(a)}\right)$$

which takes approximate form:

$$|\vec{p}| \simeq p_0\left(1 - b_1\frac{p_0}{\kappa} + b_2\frac{p_0^2}{\kappa}\right)$$

Deformed dispersion relation leads to time delay :

$$\Delta t \simeq -\frac{l}{c}\frac{p_0}{\kappa}\left(2b_1 - 3b_2\frac{p_0}{\kappa}\right)$$

where l is a distance from the source of high energy photons.

Jordanian one-parameter family of Drinfeld twists

.

The time delay for photons is:

$$\Delta t \simeq -\frac{l}{c}\frac{p_0}{\kappa}\left(-(1+r) - (1+3r+2r^2)\frac{p_0}{2\kappa}\right) =$$

$$= \frac{l}{c}\frac{p_0}{\kappa}\left(1+r+(1+3r+2r^2)\frac{p_0}{2\kappa}\right)$$

Time delay from Abelian twists

$$\Delta t = -\frac{l}{c}\frac{|\vec{p}|}{\kappa}\left(2s - 1 + \frac{|\vec{p}|}{2\kappa}s(s-1)\right)$$

Standard version of DSR : $s = 1$

As an example let us choose : $\Psi = exp(-hp_0), \quad \Gamma = exp(-hp_0)$. Then the representation of the Poincaré Lie algebra in this Hilbert space has the form:

$$M_i = \frac{1}{2}\epsilon_{ijm}(x_j p_m - x_m p_j)$$

$$N_i = \frac{1}{2h}x_i \left(e^{-2hp_0} - 1\right) + x_0 p_i + \frac{ih}{2}x_i \vec{p}^2 + h\, x^k p_k\, p_i$$

$$P_i = p_i e^{hp_0}, \qquad P_0 = h^{-1}\sinh(hp_0) + \frac{h}{2}\vec{p}^2 e^{hp_0}$$

Moreover, one can easily see that the operators (M_i, N_i, p_μ) constitute **the bicrossproduct basis**. Therefore, dispersion relation expressed in terms of the canonical momenta p_μ recovers the standard version of doubly special relativity theory.

$\mathcal{C}_h = h^{-2}(e^{-\frac{1}{2}hp_0} - e^{\frac{1}{2}hp_0})^2 - \vec{p}^2 e^{hp_0}$ implies

$m_\kappa^2 = [2h^{-1}\sinh(\frac{hp_0}{2})]^2 - \vec{p}^2 e^{hp_0}, \quad \Delta t = -\frac{l}{c}\frac{|\vec{p}|}{\kappa}$

3 Canonical Kappa- Minkowski spacetime

as $q-$analog κ-Poincaré Hopf module algebra

Motivation for $q-$analog κ-Poincaré Hopf module algebra

Intro:

1. "q-analog" version allows us to fix the value of the parameter κ.
2. This new approach is based on reformulation of Hopf algebra in such a way to cancel infinite series, this allows us to obtain one parameter family of isomorphic Hopf algebras enumerated by numerical parameter κ.
3. From mathematical point of view, this means that parameter κ is irrelevant. However from physical point of view value of κ depends on system of units we are working in, one can use natural (Planck) system of units $\hbar = c = \kappa = 1$ without changing mathematical properties of the underlying quantum model.

This is so-called **"specialization" procedure** .

"q-analog" of $\kappa-$Poincaré algebra

The idea is to introduce two group-like elements Π_0, Π_0^{-1}:$\Pi_0^{-1}\Pi_0 = 1$.
Π_0 and Π_0^{-1} are considered mutually inverse.
Now a universal, unital associative algebra generated by eleven generators $(M_i, N_i, P_i, \Pi_0, \Pi_0^{-1})$ has the following set of commutation relations:

$$\Pi_0^{-1}\Pi_0 = \Pi_0\Pi_0^{-1} = 1, \quad [P_i, \Pi_0] = [M_j, \Pi_0] = 0, \quad [N_i, \Pi_0] = -\frac{\imath}{\kappa}P_i$$

$$[N_i, P_j] = -\frac{\imath}{2}\delta_{ij}\left(\kappa(\Pi_0 - \Pi_0^{-1}) + \frac{1}{\kappa}\vec{P}^2\Pi_0^{-1}\right)$$

remaining ones between (M_i, N_i, P_i) are the same as in the Poincare Lie algebra.

The quantum algebra structure $\mathcal{U}_\kappa(\mathfrak{io}(1,3))$ is provided by defining coproduct, antipode and counit, i.e. a Hopf algebra structure. We set

$$\Delta\left(M_i\right) = M_i \otimes 1 + 1 \otimes M_i \quad \Delta\left(N_i\right) = N_i \otimes 1 + \Pi_0^{-1} \otimes N_i - \frac{1}{\kappa}\epsilon_{ijm} P_j \Pi_0^{-1} \otimes M_m$$

$$\Delta\left(P_i\right) = P_i \otimes \Pi_0 + 1 \otimes P_i \qquad \Delta(\Pi_0) = \Pi_0 \otimes \Pi_0, \quad \Delta(\Pi_0^{-1}) = \Pi_0^{-1} \otimes \Pi_0^{-1}$$

and the antipodes.

Corresponding version of κ -Minkowski spacetime (canonical κ -Minkowski spacetime) is universal enveloping algebra of solvable Lie algebra without h-adic topology.

In this version $\kappa-$Minkowski spacetime commutation relations have the form:

$$[X^0, X^i] = \frac{\imath}{\kappa} X^i; \qquad [X^i, X^j] = 0$$

Now we assume that $\kappa \in \mathbb{C}^*$ takes numerical value (specialization), which could be identified with $\kappa = M_Q$ - quantum gravity scale. Above relations define solvable Lie algebra $\mathfrak{an}^3$. Take $\mathcal{U}_{\mathfrak{an}^3} = \mathcal{M}^4_\kappa$ to be usual enveloping algebra of $\mathfrak{an}^3$, i.e.

$$\mathcal{M}^4_\kappa = \mathcal{U}_{\mathfrak{an}^3} = \frac{T\mathfrak{an}^3}{J}$$

J and ideal generated by those relations. And this introduces

canonical $\kappa-$Minkowski spacetime $\mathcal{M}^4_\kappa$ as universal enveloping algebra of solvable Lie algebra without dependence on κ

To assure that canonical $\kappa-$Minkowski algebra $\mathcal{M}^4_\kappa$ in this case is a $\mathcal{U}_\kappa(\mathfrak{io}(1,3))$ (Hopf) module algebra one has to check consistency conditions. Then it can be used in crossed product construction.

$\mathcal{U}_\kappa(\mathfrak{io}(1,3))$ acts covariantly on $\mathcal{M}^4_\kappa$:

$P_i \triangleright X^\nu = -i\delta_i^\nu$; $\qquad M_{\mu\nu} \triangleright X^\rho = -iX_\mu \delta_\nu^\rho + iX_\nu \delta_\mu^\rho$;

$\Pi_0^\pm \triangleright X^\mu = X^\mu \mp \imath\kappa^{-1}\delta_0^\mu$.

Canonical DSR algebra $\mathcal{M}^4_\kappa \rtimes \mathcal{U}_\kappa(\mathfrak{io}(1,3))$

$$
\begin{aligned}
&[M_i, X_0] = 0 && [N_i, X_0] = -\imath X_i - \tfrac{\imath}{\kappa} N_i \\
&[M_i, X_j] = \imath\epsilon_{ijl} X_l && [N_i, X_j] = -\imath\delta_{ij} X_0 + \tfrac{\imath}{\kappa}\epsilon_{ijl} M_l \\
&[P_k, X_0] = 0, && [P_k, X_j] = -\imath\delta_{jk}\Pi_0 \\
&[X_i, \Pi_0] = 0, && [X_0, \Pi_0] = -\tfrac{\imath}{\kappa}\Pi_0
\end{aligned}
$$

Notice:

- "q-analog" version of DSR algebra is not a pseudo-deformation type as the one introduced in "h-adic" case.
- The most interesting subalgebra of $\mathcal{M}^4_\kappa \rtimes \mathcal{U}_\kappa(\mathfrak{io}(1,3))$ is canonical Weyl algebra $\mathfrak{W}^4_C$ (canonical phase space) given by last two lines of above relations and κ- Minkowski algebra.

Representation in the Hilbert space

Let us first introduce operator realization of analytic function

$$\check{f} = \int f(t) dE_\kappa(t)$$

as a spectral integral with spectral measure $dE_\kappa(t)$ corresponding to self-adjoint operator $E_\kappa(t) = -\frac{\imath}{\kappa}\partial_0$, where $\kappa \in \mathbb{R}^*$. Notice that $E_1 = p_0$.

One can introduce Heisenberg representation in the Hilbert space $\mathcal{L}^2(\mathbb{R}^4, dx^4)$ of $\mathcal{M}_\kappa^4 \rtimes \mathcal{U}_{\kappa(\mathfrak{io}(1,3))}$-Canonical DSR algebra realization (non-covariant one).

- Then Hilbert space (Heisenberg) representation of DSR algebra generators has the form (functions Ψ, Γ are the same as before):

$$X^i = x^i \check{\Gamma}\check{\Psi}^{-1}, \quad X^0 = x^0 \check{\psi} - h x^k p_k \check{\gamma};$$

$$\Pi_0 = \check{\Psi}^{-1}; \qquad P_i = p_i \check{\Gamma}^{-1}$$

as Heisenberg realization of Weyl algebra (phase space)

- and the rest of generators

$$M_i = \imath \epsilon_{ijk} x_j p_k$$

$$N_i = \kappa x_i \check{\Gamma} \frac{\check{\Psi}^{-1} - \check{\Psi}}{2} - x_0 p_i \check{\psi} \check{\Psi} \check{\Gamma}^{-1} + \frac{\imath}{2\kappa} x_i \vec{p}^2 \check{\Psi} \check{\Gamma}^{-1} - \frac{1}{\kappa} x^k p_k p_i \check{\gamma} \check{\Psi} \check{\Gamma}^{-1}$$

- with Casimir operator as: $\mathcal{C}_\kappa = \kappa^2(\check{\Psi}^{-1} + \check{\Psi} - 2) - \vec{p}^2 \check{\Psi} \check{\Gamma}^{-2}$

Here $p_\mu = -\imath \partial_\mu$ and x^ν are self-adjoint operators acting in Hilbert space $\mathcal{L}^2(\mathbb{R}^4, dx^4)$. This leads to the Stückelberg version of relativistic Quantum Mechanics.

Quantization of relativistic symplectic structure

Alternatively, one can consider relativistic symplectic structure

$$\{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = 0, \qquad \{x^\mu, p_\nu\} = \delta^\mu_\nu$$

determined by the symplectic two-form $\omega = dx^\mu \wedge dp_\mu$ on the phase space $\mathbb{R}^4 \times \mathbb{R}^4$. Now we can interpret formulas for the fixed value $h = \frac{1}{\kappa}$ as a non-canonical transformation (change of variables) in the phase space. Thus in this new variables one gets $\kappa-$deformed phase space with deformed Poisson brackets replacing the commutators $([P_\mu, \mathcal{X}_\nu])$: $\{\ ,\ \}_\kappa = \frac{1}{\imath}[\ ,\ ]$. Therefore, the operators stand for true (Hilbert space) quantization of this deformed symplectic structure.

Magueijo-Smolin model

As a yet another example let us consider deformed phase space of Magueijo-Smolin model:

$$\{X^\mu, X^\nu\} = \frac{1}{\kappa}(a^\mu X^\nu - a^\nu X^\mu)$$

$$\{P_\mu, P_\nu\} = 0, \qquad \{X^\mu, P_\nu\} = \delta^\mu_\nu + \frac{1}{\kappa} a^\mu P_\nu$$

It corresponds to the following change of variable in the phase space:

$$X^\mu = x^\mu - \frac{a^\mu}{\kappa} x^\nu p_\nu, \qquad P_\mu = p_\mu$$

We do not know twist realization for this algebra.

Conclusions

From the mathematical point of view we discuss two main cases (two mathematically different models of

κ-Poincare and κ-Minkowski:

h-adic

q-analog

h-adic DSR algebra

Canonical DSR algebra

- However we also point out that one should be aware that only a "q-analog" version of κ-Poincare/κ-Minkowski has to be considered if one wants to discuss physics.
- Moreover we have shown that the deformed algebra is a pseudo-deformation of an undeformed one, in a sense of change of generators.
- it is only possible after either h-adic extension of universal enveloping algebra or by introducing additional generator

From the physical point of view we focus only on one model of

κ-Poincare and κ-Minkowski:

q-analog

Canonical DSR algebra

- In this case κ - Minkowski algebra is an universal envelope of solvable Lie algebra without h-adic topology.
- This version allows us to connect the parameter κ with some physical constant, like, e.g., **quantum gravity scale** or **Planck mass** and all the realizations might have physical interpretation.
- Together with this connection a physical interpretation for deformation parameter κ, as second invariant scale, appears naturally within DSR theory and allows us to interpret deformed dispersion relations as valid at the "κ-scale" (as Planck scale or Quantum Gravity scale) when quantum gravity corrections become relevant.

- Realizations of DSR algebras lead to physically different models of DSR theory.
- What is important in our approach that it is always possible to choose physical frame (physically measurable momenta and position) by undeformed Weyl algebra which makes clear physical interpretation of such DSR theories. This implies that various realizations of DSR algebras are written in terms of the standard (undeformed) Weyl-Heisenberg algebra .
- Heisenberg representation in Hilbert space is provided.
- Realizations of deformed phase spaces (deformed Weyl algebra) contributes to Phase-Space-Algebra approach to DSR
.

This implies that various realizations of DSR algebras are written in terms of the standard (undeformed) Weyl-Heisenberg algebra which opens the way for quantum mechanical interpretation DSR theories in a more similar way to (proper-time) relativistic (Stuckelberg version) Quantum Mechanics instead (in Hilbert space representations contexts).

Open question

With this interpretation one can go further and ask if deformed special relativity is a quantization of doubly special relativity.