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N-symmetric Chebyshev polynomials in "compound model" for generalized oscillator [1]

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1. Introduction. The Jacobi matrices appeared in studies of "compound model" of generalized oscillator have the following form

$$J = [a_{i,\,j}]_{i,\,j=0}^{\infty}\,, \qquad a_{i,\,j} = a_i\delta_{i,\,j} + \delta_{i+1,\,j} + \delta_{i-1,\,j}, \tag{1}$$

where $a_{i+N} = a_i, \quad a_i \in \mathbb{C}, \quad N \in \mathbb{N}$. The polynomials $\{\psi_n(x)\}_{n=0}^{\infty}$ related to J fulfill the recurrence relations

$$x\psi_n(x) = \psi_{n+1}(x) + a_n\psi_n(x) + \psi_{n-1}(x), \quad \psi_0(x) = 1,\ \psi_{-1}(x) = 0. \tag{2}$$

Because matrix J differs from Jacobi matrix for Chebyshev polynomials of the 2-nd kind only by presence of nonzero main diagonal the polynomials $\psi_n(x)$ are called "nonstandard" Chebyshev polynomials.

If in addition polynomials $\psi_n(x)$ fulfill the recurrence relations

$$x^N\psi_{Nm+k} = B_m^{(N)}\psi_{N(m+1)+k} + A_m^{(N)}\psi_{Nm+k} + B_{m-1}^{(N)}\psi_{N(m-1)+k}, \tag{3}$$

for all $k = \overline{0, (N-1)}$ and all m such that

$$1)\ m \geq 0, \qquad \text{or} \quad 2)\ m \geq 1 \ (\text{but not for}\, m = 0),$$

then they are called N-symmetric Chebyshev polynomials.

The case 1) was considered early for $N = 2, 3$. In this talk we consider the case 2). Namely we construct N-symmetric Chebyshev polynomials for $N = 3, 4, 5$. The existence of such polynomials for $N > 5$ is still an open question.

2. The case $N = 3$. In this case Jacobi matrix has the form

$$J = \begin{bmatrix} A_1 & A_2 & 0 & 0 & 0 & \cdots \\ A_3 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & A_3 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & A_3 & A_1 & A_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \qquad \text{where}$$

$$A_1 = \begin{bmatrix} i\sqrt{3} & 1 & 0 \\ 1 & -i\sqrt{3} & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The recurrence relations (3) for $m \geq 1$ takes the form

$$x^3\psi_{3m+k}(x) = \psi_{3(m+1)+k}(x) + \psi_{3(m-1)+k}(x), \quad k = 0, 1, 2,$$

and for $m = 0$

$$\begin{aligned}
x^3\psi_0(x) &= \psi_3(x) - \psi_1(x) + 2i\sqrt{3}\psi_0(x), \\
x^3\psi_1(x) &= \psi_4(x) - \psi_0(x), \\
x^3\psi_2(x) &= \psi_5(x).
\end{aligned}$$

First six polynomials are

$$\begin{aligned}
\psi_0(x) &= 1, & \psi_1(x) &= x - i\sqrt{3}, \\
\psi_2(x) &= x^2 + 2, & \psi_3(x) &= x^3 + x + i\sqrt{3}, \\
\psi_4(x) &= x^4 - i\sqrt{3}x^3 + 1, & \psi_5(x) &= x^5 + 2x^2,
\end{aligned}$$

and for $n \geq 6$ $\qquad \psi_n(x) = x^3\psi_{n-3}(x) - \psi_{n-6}(x).$

The above polynomials are orthogonal in a sense

$$(\psi_m(x), \psi_n(x))_{\mathrm{L}^2(\mathcal{D};\mu} = (-1)^n \delta_{m,n}$$

with respect to complex-valued measure $\mu = \mu_{disc} + \mu_{cont}$. The support $\mathcal{D}$ of this measure are given on the Fig.1.

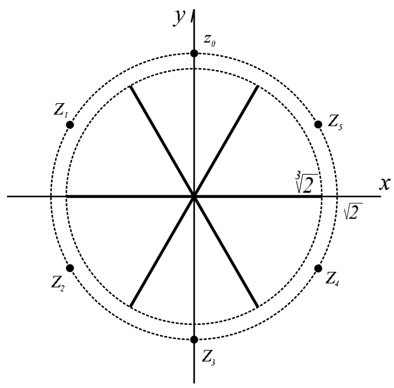

Figure 1: Support $\mathcal{D}$ of the measure μ

The discrete part μ_{disc} of the measure μ is concentrated at the points

$$z_p = \sqrt{2}\, e^{i(\frac{\pi}{2}+p\frac{\pi}{3})},\ p = \overline{0,5} \quad \text{with loads} \quad \sigma_p = \frac{1}{\sqrt{6}},\ (p \neq 3); \quad \sigma_3 = -\frac{2}{\sqrt{6}}.$$

The continuous part μ_{cont} of the measure μ is concentrated in the set of rays

$$\mathbb{R}^{(3)} = \bigcup R_{\alpha+2k\pi/3}, \quad k = 0, 1, 2;$$

where by R_β we denote the ray, passing through the origin of the complex plane at an angle β to the real axis.

The density function $f(y), \quad x = e^{i\alpha}y$, of μ_{cont} defined on the ray R_α with $\alpha = 0, \pm\frac{2\pi}{3}$ is equal to

$$f(y) = \left(-1 + \frac{2}{y^2}e^{-2\alpha i} + \frac{y^2}{2}e^{2\alpha i}\right) \frac{y^5}{\pi(8+y^6)} \sqrt{\frac{4-y^6}{y^6}}\, \chi_{[-\sqrt[3]{2}, \sqrt[3]{2}]}(y),$$

where $\chi_{[a,b]}$ is the characteristic function of the segment $[a, b]$.

The moments of the measure μ equals to

$$\mu_{2n+1} = i(-2)^n\sqrt{3}, \quad n \geq 0;$$

$$\mu_{2n} = \mu_{2(3k+j)} = (-2)^n + (-2)^{j-2}\left(1-\delta_{k,0}\right)\sum_{s=0}^{k-1}(-2)^{3(k-s)}c_s + c_k\delta_{j,2},$$

where $n = 3k+j$, $k \geq 0$, $j = 0,1,2$, and $c_s = \frac{(2s)!}{s!(s+1)!}$ are Catalan numbers.

Note that N-symmetric polynomials splits into N series of polynomials. In considered case ($N = 3$) we have three series

$$\{\psi_n(x)\}_{n=0}^{\infty} = \bigcup_{k=0}^{2}\left[\{\psi_{3m+k}(x)\}_{m=0}^{\infty}\right].$$

We find that

1) Differential equations for polynomials are distinct for different series. These equations are of 2-nd order with different polynomial coefficients; In the case $N = 3$ the differential equation for $y_m^{(k)} = \psi_{3m+k}(x), \quad M \geq 0, \; k = 0, 1, 2,$ has the form

$$A_m^{(k)} y_m^{(k)\prime\prime}(x) + B_m^{(k)} y_m^{(k)\prime}(x) + C_m^{(k)} y_m^{(k)}(x) = 0, \tag{6}$$

where $A_m^{(k)}$, $B_m^{(k)}$, $C_m^{(k)}$ are polynomials in x of 11, 10 and 9 degree respectively, given by

$$A_m^{(0)} = -x(4x-6)\left[(x+i\sqrt{3})(x^3+x+i\sqrt{3})+\right.$$
$$\left.+\frac{x^3}{6m}(2x+3i\sqrt{3})+\frac{3(m+1)x^2+2}{3mx^2}\right];$$
$$B_m^{(0)} = \left(\frac{m+1}{2m}x^3+x+i\sqrt{3}\right)\left[(8+7x^6)(\frac{1}{2}x^3+x+i\sqrt{3})+\right.$$
$$\left.+(4-x^6)(2+\frac{3}{2}(m+2)x^2)\right]+\frac{4-x^6}{6mx^2}\left[\frac{3}{2}(8+7x^6)(m+1)x^2-\right.$$
$$\left.-9x^5(m+1)\left(\frac{m+2}{2}x^3+(m-1)(x+i\sqrt{3})\right)\right];$$
$$C_m^{(0)} = (\frac{1}{2}x^3+x+i\sqrt{3})\left[\frac{3}{2}(8+7x^6)(m+1)x^2-\right.$$
$$\left.-9x^5\left(\frac{1}{2}(m+1)(m+2)x^3+(m^2-1)(x+i\sqrt{3})\right)\right]+$$
$$+(1+\frac{3}{2}(m+1)x^2)\left[(8+7x^6)(\frac{1}{2}x^3+x+i\sqrt{3})+\right.$$
$$\left.+(4-x^6)(2x+\frac{3}{2}(m+2)x^3\right].$$

$$A_m^{(1)}=x(4-x^6)\left[\frac{4-x^6}{6mx^2}-1-(x-i\sqrt{3})\left(\frac{2m+1}{2m}x^3+\frac{m+1}{m}(x-i\sqrt{3})\right)\right];$$

$$B_m^{(1)}=(8+7x^6)\left(1+\frac{2m+1}{2m}(x-i\sqrt{3})x^3+\frac{m+1}{m}(x-i\sqrt{3})^2\right)+$$
$$+\frac{4-x^6}{2}\left[2x^4+\frac{x-i\sqrt{3}}{m}(3(2m+1)x^3+4(m+1)x\right];$$

$$C_m^{(1)}=\frac{m+1}{2}\left[\frac{8+7x^6}{m}(x^3+2(x-i\sqrt{3}))+\frac{x(4-x^6)}{m}(3(3m-1)x^2-4)-\right.$$
$$\left.-18x^5\left(m-1+\frac{2m+1}{2}(x-i\sqrt{3})x^3+(m+2)(x-i\sqrt{3})^2\right)\right].$$

$$A_m^{(2)}=x(4-x^6)(x^2+2)^2;$$
$$B_m^{(2)}=-(x^2+2)(3x^8+14x^6+24x^2+16);$$
$$C_m^{(2)}=8x(x^8+4x^6+5x^2+2)+9x^5m(m+2)(x^2+2)^2.$$

To obtain the diff.eq. from (6) with $k=0,1$, and $m=0$ it is need first to multiply relation (6) by m and then equate m to zero.

2) Each of this series related to some generalized oscillator — "elementary" Chebyshev oscillator. We find the connection between representations of the algebra of 3-symmetric Chebyshev oscillator and representations of three algebras related to "elementary" Chebyshev oscillators. This is just the compound model of generalized 3-symmetrical Chebyshev oscillator.

3) Hopefully we can prove that the set $\mathcal{D}$ is the set of accumulation points of zeros for polynomials $\{\psi_n(x)\}_{n=0}^{\infty}$, but up to now we proved this result only for the series with $k = 2$.

3. The support of the continuous part μ_{cont} for the orthogonality measure μ.

It is known that investigation of the spectrum for the periodic Jacobi matrix

$$J = [a_{i,\,j}]_{i,\,j=0}^{\infty}\,, \qquad a_{i,\,j} = a_i\delta_{i,\,j} + \delta_{i+1,\,j} + \delta_{i-1,\,j},$$

(where $a_{i+N} = a_i, \quad a_i \in \mathbb{C}, \quad N \in \mathbb{N}$) is reduced to finding the eigenvalues $\mu_{1,2}$ of the matrix

$$T = T(\lambda) = \begin{bmatrix} 0 & 1 \\ -1 & \lambda - a_{N-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ -1 & \lambda - a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & \lambda - a_0 \end{bmatrix}.$$

It is clear that $\det T = 1$ and $\sigma(T) = \{\mu_1, \mu_2\}$ with $\mu_1 \cdot \mu_2 = 1$ whenever it follows that $|\mu_1|\,|\mu_2| = 1$.

There are two variants:

1) $|\mu_1| < 1,\ |\mu_2| > 1$ — this is, as it is called, "hyperbolic" case. In this case the matrix J has not continuous spectrum but only eigenvalues of finite multiplicity.

2) $|\mu_1| = |\mu_2| = 1$ — this is what is known as "elliptic" case. In this case the matrix J has only a continuous spectrum the boundary points of which are defined by condition $|\mu_1| = |\mu_2| = 1$.

In this talk we consider only the "elliptic" case, that is we describe only support of μ_{cont}.

Eigenvalues $\mu_{1,2}$ for T are the roots of the equation

$$\det\left(T - \mu\mathbb{I}\right) = 0 \quad \text{or} \quad \mu^2 - \operatorname{Tr} T\,\mu + 1 = 0.$$

Then

$$\mu_{1,2} = \frac{1}{2}\mathrm{Tr}T \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2 T - \det T}.$$

If we denote

$$\mu_1 = e^{i\varphi_1}, \quad \mu_2 = e^{i\varphi_2}, \quad z = \frac{1}{2}\mathrm{Tr}\, T$$

then we obtain

$$z = \frac{1}{2}\left(\mu_1 + \mu_2\right) = \frac{1}{2}\left(e^{i\varphi_1} + e^{i\varphi_2}\right)$$

Now we have

$$\frac{1}{4}\mathrm{Tr}^2\, T - 1 = z^2 - 1 = (\mu_1 - z)^2 \;\Rightarrow\; -1 = -e^{i(\varphi_1+\varphi_2)}.$$

and hence,

$$\varphi_1 + \varphi_2 = 0, \quad z = \frac{1}{2}\mathrm{Tr}\, T = \cos\varphi_1. \tag{7}$$

A. $N=3$. We have $\operatorname{Tr} T = \lambda^3$ and from (7) it follows that

$$\lambda^3 = 2\cos\varphi_1 .$$

Thus the maximal value of $\left|\lambda^3\right|$ is obtained when $\varphi_1 = 0; \pi$. It gives the boundary for continuous part of the spectrum

$$\lambda_k = \sqrt[3]{2} e^{i2k\pi/3}, \quad \widetilde{\lambda}_k = \sqrt[3]{2} e^{i(2k+1)\pi/3}, \quad k = 0, 1, 2.$$

We obtain the support of μ_{cont} given in the Fig.1 above.

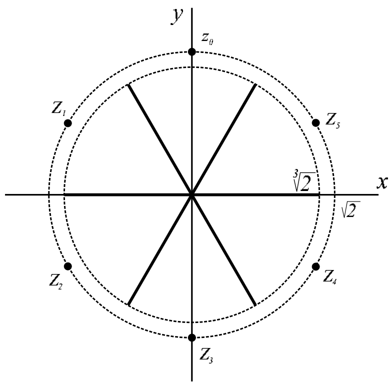

Figure 2: Support $\mathcal{D}$ of the measure μ once more

B. $N = 4$. Jacobi matrix looks as

$$J = \begin{bmatrix} A_1 & A_2 & 0 & 0 & 0 & \cdots \\ A_3 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & A_3 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & A_3 & A_1 & A_2 & \cdots \\ \multicolumn{6}{c}{\dotfill} \end{bmatrix}, \qquad \text{where}$$

$$A_1 = \begin{bmatrix} 2i & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2i & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Related recurrent relations (3) for $m \geq 1$ takes the form

$$x^4 \psi_{4m+k}(x) = \psi_{4(m+1)+k}(x) - 2\psi_{4m+k}(x) + \psi_{4(m-1)+k}(x),$$

for all $k = 0, 1, \ldots, N-1$;

For $m = 0$ they looks as

$$x^4\psi_0(x) = \psi_4(x) - \psi_2(x) - 4i\psi_1(x) + 6\psi_0(x),$$
$$x^4\psi_1(x) = \psi_5(x) - 3\psi_1(x) - 4i\psi_0(x),$$
$$x^4\psi_2(x) = \psi_6(x) - 2\psi_2(x) - \psi_0(x),$$
$$x^4\psi_3(x) = \psi_7(x) - 2\psi_3(x).$$

First eight polynomials are

$$\psi_0(x) = 1; \quad \psi_1(x) = x - 2i;$$
$$\psi_2(x) = x^2 - 2ix - 1; \quad \psi_3(x) = x^3 + -2x;$$
$$\psi_4(x) = x^4 + x^2 + 2ix + 1;$$
$$\psi_5(x) = x^5 - 2ix^4 + 3x - 2i;$$
$$\psi_6(x) = x^6 - 2ix^5 - x^4 + 2x^2 - 4ix - 1;$$
$$\psi_7(x) = x^7 + 2x^5 + 2x^3 + 2ix^2 + 4x.$$

For $n \geq 8$ we have

$$\psi_n(x) = (x^n + 2)\psi_{n-4}(x) - \psi_{n-8}(x).$$

Let us find the eigenvalues μ_i for matrix $T = T(\lambda)$. In the case $N = 4$ we have

$$a_0 = 2i,\ a_1 = 0,\ a_2 = -2i,\ a_3 = 0, \quad \mathrm{Tr} T = 2 + \lambda^4.$$

From relation (7) have

$$z = \frac{2+\lambda^4}{2} = \cos\varphi_1 \quad \Rightarrow \quad \lambda^4 = -4\sin^2\varphi_1.$$

Taking into account that maximum value of $\sin^2\varphi_1$ is equal to 1, we find the boundaries of the continuous spectrum of Jacobi matrix J solving the equation $\lambda^4 = 4e^{i\pi}$. This gives

$$\lambda_k = \sqrt{2}e^{\frac{\pi+2k\pi}{4}}, \quad k = 0, 1, 2, 3.$$

Hence the support of the continuous spectrum is given on the Fig.2.

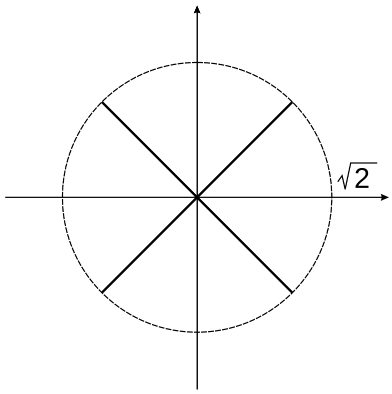

Figure 3: Support $\mathcal{D}$ of the measure μ_{cont} N=4

C. $N=5$. The Jacobi matrix is

$$J=\begin{bmatrix} A_1 & A_2 & 0 & 0 & 0 & \cdots \\ A_3 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & A_3 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & A_3 & A_1 & A_2 & \cdots \\ \multicolumn{6}{c}{\dotfill} \end{bmatrix}, \qquad \text{where}$$

$$A_1=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & i\sqrt{5} & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 01 & -i\sqrt{5} \end{bmatrix}, A_2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, A_3=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The recurrence relations (3) for $m \geq 1$ are

$$x^5\psi_{5m+k}(x) = \psi_{5(m+1)+k}(x) - i\sqrt{5}\psi_{4m+k}(x) + \psi_{5(m-1)+k}(x), \quad (10)$$

for all $k = \overline{0;4}$ and for $m = 0$ are

$$x^5\psi_0(x) = \psi_5(x) - \psi_3(x) - i\sqrt{5}\psi_0(x),$$
$$x^5\psi_1(x) = \psi_6(x) - \psi_2(x) - i\sqrt{5}\psi_1(x),$$
$$x^5\psi_2(x) = \psi_7(x) - \psi_1(x),$$
$$x^5\psi_3(x) = \psi_8(x) - \psi_0(x),$$
$$x^5\psi_4(x) = \psi_9(x).$$

First ten polynomials looks as

$$\begin{aligned}
&\psi_0(x) = 1; \quad \psi_1(x) = x;\\
&\psi_2(x) = x^2 - i\sqrt{5}x - 1;\\
&\psi_3(x) = x^3 - i\sqrt{5}x^2 - 2x;\\
&\psi_4(x) = x^4 - i\sqrt{5}x^3 - 3x^2 + i\sqrt{5}x + 1;\\
&\psi_5(x) = x^5 + x^3 - i\sqrt{5}x^2 - 2x + i\sqrt{5};\\
&\psi_6(x) = x^6 + x^2 - 1;\\
&\psi_7(x) = x^7 - i\sqrt{5}x^6 - x^5 + x;\\
&\psi_8(x) = x^8 - i\sqrt{5}x^7 - 2x^6 + 1;\\
&\psi_9(x) = x^9 - i\sqrt{5}x^8 - 3x^7 + i\sqrt{5}x^6 + x^5.
\end{aligned}$$

Note that for $n \geq 10$ the recurrent relations can be written as

$$\psi_n(x) = x^5\psi_{n-5}(x) - \psi_{n-10}(x).$$

From the relation $\mathrm{Tr}T = \lambda^5$ we obtain as before $\lambda^5 = 2\cos\varphi_1$.Hence, the maximal value $\left|\lambda^5\right|$ is obtained for $\varphi_1 = 0, \pi$. This gives the boundaries of continuous spectrum for Jacobi matrix J

$$\lambda_k = \sqrt[5]{2}\, e^{i2k\pi/5}, \quad \widetilde{\lambda}_k = \sqrt[5]{2}\, e^{i(\pi+2k\pi)/5}, \quad k = \overline{0;4}.$$

Thus the support looks as shown on the Fig.3

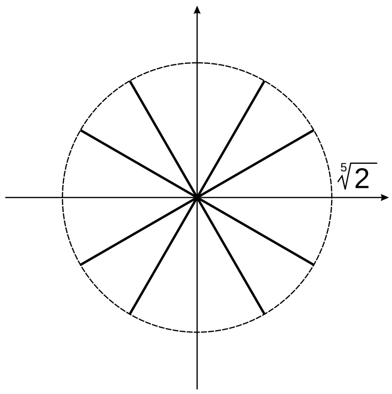

Figure 4: Support $\mathcal{D}$ of the measure μ_{cont} N=5

Remark. The investigation of the discrete part μ_{disc} for $N = 4$ and $N = 5$ is not finished and we can only say that the support for μ_{disc} is concentrated at eight ($N = 4$) or ten ($N = 5$) points which lies on the circles having the center in the origin.