Factorization of R-matrix, Baxter Q-operators and SOV

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motivation: Separation of Variables

Let $\Psi(\mathbf{z}) = \Psi(z_1, \cdots, z_N)$ is eigenfunction of a quantum integrable system
Expand $\Psi(z_1, \cdots, z_N)$ into a basis of eigenfunctions of another(auxiliary) integrable system

$$\Psi(\mathbf{z}) = \int \mathrm{U}(\mathbf{z}, \mathbf{x})\, \mathrm{Q}(\mathbf{x})\, \mu(\mathbf{x})\, \mathrm{d}^N \mathbf{x}$$

Separation of variables $\leftrightarrow$ factorization of $\mathrm{Q}(\mathbf{x})$: $\mathrm{Q}(\mathbf{x}) = \mathrm{Q}(x_1)\mathrm{Q}(x_2)\cdots\mathrm{Q}(x_N)$

- ✔ $\mathrm{U}(\mathbf{z}, \mathbf{x})$ can be constructed iteratively and $\mathrm{Q}(x)$ is solution of Baxter equation [Sklyanin]

main example

- ✔ integrable system: XXX spin chain $S_k = z_k\partial_k + \ell\,;\; S_k^+ = z_k^2\partial_k + 2\ell z_k\,; S_k^- = -\partial_k$

$$\mathrm{L}_k(u) = \begin{pmatrix} u + S_k & S_k^- \\ S_k^+ & u - S_k \end{pmatrix} \;;\; T(u) = L_1(u)\cdots L_N(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$\mathrm{t}(u) = \mathrm{tr}\, T(u) = A(u) + D(u) = 2\cdot u^N + u^{N-2}\cdot \mathbf{q}_2 + u^{N-3}\cdot \mathbf{q}_3 + \cdots + \mathbf{q}_N \quad;\quad [\mathrm{t}(u), \mathrm{t}(v)] = 0$$

$$\mathbf{q}_k \Psi(z_1, \cdots, z_N) = q_k \Psi(z_1, \cdots, z_N)$$

- ✔ auxiliary integrable system $[\mathrm{B}(u), \mathrm{B}(v)] = 0 \quad;\quad \mathbf{x} = x_1, \cdots, x_{N-1}$

$$\mathrm{B}(u)\mathrm{U}(\mathbf{z}, \mathbf{x}, p) = p(u - x_1)\cdots(u - x_{N-1})\cdot \mathrm{U}(\mathbf{z}, \mathbf{x}, p) \quad;\quad \mathrm{S}^-\mathrm{U}(\mathbf{z}, \mathbf{x}, p) = p\cdot \mathrm{U}(\mathbf{z}, \mathbf{x}, p)$$

- ✔ Baxter equation

$$\mathrm{t}(x)\mathrm{Q}(x) = \Delta_+(x)\cdot \mathrm{Q}(x+1) + \Delta_-(x)\cdot \mathrm{Q}(x-1)$$

examples and plan

Q-operators $\leftrightarrow$ R.Baxter

Baxter, Bazhanov, Stroganov, Lukyanov, Zamolodchikov, Volkov, Faddeev, Kashaev, Pasquier, Gaudin, Pronko, Smirnov, Korff, Sklyanin, Kuznetsov, Salerno, SD, Korchemsky, Manashov, Kulish, Kharchev, Lebedev, Gerasimov, Oblezin, Bytsko, Teschner ...

SOV:

- ✔ quantum Toda chain: Gutzwiller, Sklyanin, Pasquier-Gaudin, Kharchev, Lebedev, Gerasimov, Oblezin, Silantyev
- ✔ XXX spin chain: Faddeev, Korchemsky, SD, Manashov, Lipatov
- ✔ q-deformed XXX spin chain: Manashov, Kirch, Bytsko, Teschner
- ✔ SL(n)-invariant spin chain: Sklyanin, Smirnov

Plan: algebraic part – demonstration of the connection: R - matrix $\leftrightarrow$ $\mathrm{U}(\mathbf{z},\mathbf{x},p)$ $\leftrightarrow$ Q-operators

- ✔ principal series representation of $SL(2,C)$
- ✔ solution of general Yang-Baxter equation and its factorization
- ✔ factorization of general transfer-matrix into Q-operators
- ✔ iterative diagonalization of $\mathrm{B}(u)$ [SD,Korchemsky,Manashov]
- ✔ repetition of all steps except last ones for $SL(n,C)$ [SD,Manashov]

$$\mathrm{z} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \;;\; \mathrm{h} = \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix} \;;\; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

main construction: representation with labels (σ_1, σ_2) and $(\bar\sigma_1, \bar\sigma_2)$, $\sigma_k - \bar\sigma_k \in \mathbb{Z}$

$$g^{-1} \cdot \mathrm{z} = \mathrm{z}' \cdot \mathrm{h} \quad ; \quad \mathrm{T}(g)\,\Phi(\mathrm{z}) = [h_{11}]^{\sigma_1+1} \cdot [h_{22}]^{\sigma_2+2} \cdot \Phi(\mathrm{z}')$$

$[z]^a \equiv z^a \cdot \bar z^{\bar a} = |z|^{2a} \cdot \bar z^{\bar a - a}$, where z and $\bar z$ are complex conjugate, but $a - \bar a \in \mathbb{Z}$

explicit formulae:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-c+az}{d-bz} & 1 \end{pmatrix} \begin{pmatrix} \frac{d-bz}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & \frac{1}{d-bz} \end{pmatrix}$$

$$\mathrm{T}\,(g)\;\Phi(z,\bar z) = [ad-bc]^{1-\sigma_1} \cdot [d-bz]^{\sigma_1-\sigma_2-1} \cdot \Phi\left(\frac{-c+az}{d-bz}, \frac{-\bar c+\bar a\bar z}{\bar d-\bar b\bar z}\right)$$

intertwining operator: representations (σ_1, σ_2) and (σ_2, σ_1) are equivalent

$$\mathrm{S} \cdot \mathrm{T}^{(\sigma_1,\sigma_2)}(g) = \mathrm{T}^{(\sigma_2,\sigma_1)}(g) \cdot \mathrm{S} \quad ; \quad \mathrm{S} = (i\partial_z)^{\sigma_1-\sigma_2}\,(i\partial_{\bar z})^{\bar\sigma_1-\bar\sigma_2} = [i\partial_z]^{\sigma_1-\sigma_2}$$

representations of $GL(2, C)$

generators: $g = 1 + \epsilon \cdot \mathbf{e}_{ik}$

$$\mathrm{T}(g)\,\Phi(z,\bar{z}) = \Phi(z,\bar{z}) + \left(\epsilon \cdot \mathrm{E}_{ik} + \bar{\epsilon} \cdot \bar{\mathrm{E}}_{ik}\right)\,\Phi(z,\bar{z}) + \cdots$$

$$\mathrm{E}_{11} = z\partial_z + 1 - \sigma_1 \;;\; \mathrm{E}_{12} = z^2\partial_z + (1 - \sigma_1 + \sigma_2)z \;;\; \mathrm{E}_{21} = -\partial_z \;;\; \mathrm{E}_{22} = -z\partial_z - \sigma_2$$

commutation relations

$$[\mathrm{E}_{ik}, \mathrm{E}_{nm}] = \delta_{kn}\mathrm{E}_{im} - \delta_{im}\mathrm{E}_{nk}$$

L-operator

$$\mathrm{L}(u) = \begin{pmatrix} u + \mathrm{E}_{11} & \mathrm{E}_{21} \\ \mathrm{E}_{12} & u + \mathrm{E}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u - \sigma_1 & -\partial_z \\ 0 & u - \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$$

natural combinations $u_1 = u - \sigma_1 \;\;;\;\; u_2 = u - \sigma_2$

intertwining operator exchanges parameters in L-operator

$$\mathrm{S} = [i\partial_z]^{\sigma_{12}} = [i\partial_z]^{u_2 - u_1} \quad ; \quad [i\partial_z]^{u_2 - u_1} \cdot \mathrm{L}(u_1, u_2) = \mathrm{L}(u_2, u_1) \cdot [i\partial_z]^{u_2 - u_1}$$

principal series representations of $SL(2,C)$

$ad-bc=1 \longrightarrow$ only differences $\sigma_{12}=\sigma_1-\sigma_2$ and $\bar\sigma_{12}=\bar\sigma_1-\bar\sigma_2$ enter

$$\mathrm{T}\,(g)\;\Phi(z,\bar z)=[d-bz]^{\sigma_{12}-1}\cdot\Phi\left(\frac{-c+az}{d-bz},\frac{-\bar c+\bar a\bar z}{\bar d-\bar b\bar z}\right)$$

unitary representations: representation with labels $\sigma_{12}=-\frac{n}{2}+i\lambda\,;\;\bar\sigma_{12}=\frac{n}{2}+i\lambda\,;\;n\in\mathbb{Z}\,,\;\lambda\in\mathbb{R}$

$$\langle\Phi_1|\Phi_2\rangle=\int \mathrm{d}^2z\,\overline{\Phi_1(z,\bar z)}\,\Phi_2(z,\bar z)\;\;;\;\;\langle\mathrm{T}(g)\Phi_1|\mathrm{T}(g)\Phi_2\rangle=\langle\Phi_1|\Phi_2\rangle$$

intertwining operator is unitary

$$\mathrm{S}=(i\partial_z)^{-\frac{n}{2}+i\lambda}\,(i\partial_{\bar z})^{\frac{n}{2}+i\lambda}\longrightarrow\mathrm{S}^\dagger=(i\partial_{\bar z})^{-\frac{n}{2}-i\lambda}\,(i\partial_z)^{\frac{n}{2}-i\lambda}=\mathrm{S}^{-1}$$

symmetric labels: labels (σ_1,σ_2) and $(\sigma_1+\sigma,\sigma_2+\sigma)$ correspond the same representation of $SL(2,C)$. We fix the freedom by condition $\sigma_1+\sigma_2=1$.

Yang-Baxter equation

general Yang-Baxter equation

$$\mathrm{R}_{12}(u-v)\mathrm{R}_{13}(u)\mathrm{R}_{23}(v) = \mathrm{R}_{23}(v)\mathrm{R}_{13}(u)\mathrm{R}_{12}(u-v)$$

defining relation

$$\mathrm{R}_{12}(u-v)\mathrm{L}_1(u)\mathrm{L}_2(v) = \mathrm{L}_2(v)\mathrm{L}_1(u)\mathrm{R}_{12}(u-v)$$

it is useful to extract permutation: $\mathrm{R}_{12} = \mathrm{P}_{12}\,\check{\mathrm{R}}_{12} \quad , \quad \mathrm{P}_{12}\,\Phi(z_1, z_2) = \Phi(z_2, z_1)$

$$\check{\mathrm{R}}(u-v)\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) = \mathrm{L}_1(v_1, v_2)\,\mathrm{L}_2(u_1, u_2)\,\check{\mathrm{R}}(u-v)$$

permutation group $\mathfrak{S}_4$ of four parameters: $\mathbf{u} \equiv (v_1, v_2, u_1, u_2)$

$$s \longrightarrow \check{\mathrm{R}}(u-v)\ ;\ s(v_1, v_2, u_1, u_2) = (u_1, u_2, v_1, v_2)$$

generators of $\mathfrak{S}_4$ – elementary permutations $\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3$

$$s_1\mathbf{u} = (v_2, v_1, u_1, u_2)\ ;\ s_2\mathbf{u} = (v_1, u_1, v_2, u_2)\ ;\ s_3\mathbf{u} = (v_1, v_2, u_2, u_1)$$

Yang-Baxter equation

correspondence $s_i \to \mathrm{S}_i(\mathbf{u})$; $s_i s_j \to \mathrm{S}_i(s_j\mathbf{u})\mathrm{S}_j(\mathbf{u})$

$$(\overbrace{v_1\ ,\ v_2}^{\mathrm{S}_1}, u_1, u_2)\ :\ \mathrm{S}_1(\mathbf{u})\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_2(v_2,v_1)\mathrm{S}_1(\mathbf{u})$$

$$(v_1\,\overbrace{v_2, u_1}^{\mathrm{S}_2},\ u_2)\ :\ \mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(v_2,u_2)\,\mathrm{L}_2(v_1,u_1)\mathrm{S}_2(\mathbf{u})\,.$$

$$(v_1, v_2, \overbrace{u_1\ ,\ u_2}^{\mathrm{S}_3})\ :\ \mathrm{S}_3(\mathbf{u})\,\mathrm{L}_1(u_1,u_2) = \mathrm{L}_1(u_2,u_1)\mathrm{S}_3(\mathbf{u})$$

$$\mathrm{S}_1(\mathbf{u}) = [i\partial_2]^{v_2-v_1}\ \ ;\ \ \mathrm{S}_2(\mathbf{u}) = [z_{12}]^{u_1-v_2}\ \ ;\ \ \mathrm{S}_3(\mathbf{u}) = [i\partial_1]^{u_2-u_1}$$

defining relations for generators of permutation group $\mathfrak{S}_4$

quadratic relations are simple

$$s_1 s_1 = \mathbb{1} \to \mathrm{S}_1(s_1\mathbf{u})\mathrm{S}_1(\mathbf{u}) = [i\partial_2]^{v_2-v_1}\cdot[i\partial_2]^{v_1-v_2} = \mathbb{1}$$

qubic relations reduce to star-triangle relation $[i\partial]^a\cdot[z]^{a+b}\cdot[i\partial]^b = [z]^b\cdot[i\partial]^{a+b}\cdot[z]^a$ [Isaev]

$$s_1 s_2 s_1 = s_2 s_1 s_2 \to \mathrm{S}_1(s_2 s_1\mathbf{u})\mathrm{S}_2(s_1\mathbf{u})\mathrm{S}_1(\mathbf{u}) = \mathrm{S}_2(s_1 s_2\mathbf{u})\mathrm{S}_1(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u})$$

$$[i\partial_2]^{u_1-v_2}\cdot[z_{12}]^{u_1-v_1}\cdot[i\partial_2]^{v_2-v_1} = [z_{12}]^{v_2-v_1}\cdot[i\partial_2]^{u_1-v_1}\cdot[z_{12}]^{u_1-v_2}.$$

$$s = s_2 s_1 s_3 s_2 \to \check{\mathrm{R}}(u-v) = \mathrm{S}_2(s_1 s_3 s_2 \mathbf{u}) \mathrm{S}_1(s_3 s_2 \mathbf{u}) \mathrm{S}_3(s_2 \mathbf{u}) \mathrm{S}_2(\mathbf{u}) =$$

$$= [z_{12}]^{u_2 - v_1} \, [i\partial_2]^{u_1 - v_1} \, [i\partial_1]^{u_2 - v_2} \, [z_{12}]^{u_1 - v_2}$$

useful decomposition on permutations $u_k \leftrightarrow v_k \quad : \quad \mathfrak{S}_4 \to \mathfrak{S}_2 \times \mathfrak{S}_2$

$$s = s_2 s_1 s_2 \cdot s_2 s_3 s_2 = r_1 r_2 \to \check{\mathrm{R}}(u-v) = \mathrm{R}_1(r_2 \mathbf{u}) \mathrm{R}_2(\mathbf{u})$$

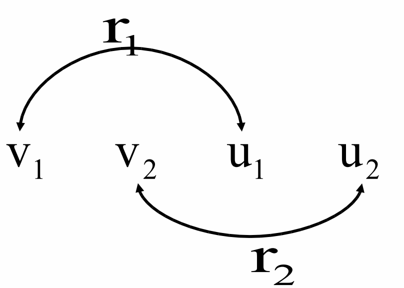
$$r_1 = s_2 s_1 s_2 \to \mathrm{R}_1(\mathbf{u}) = [z_{12}]^{v_2 - v_1} \, [i\partial_2]^{u_1 - v_1} \, [z_{12}]^{u_1 - v_2} ,$$

$$r_2 = s_2 s_3 s_2 \to \mathrm{R}_2(\mathbf{u}) = [z_{12}]^{u_2 - u_1} \, [i\partial_1]^{u_2 - v_2} \, [z_{12}]^{u_1 - v_2} .$$

defining equations

$$\mathrm{R}_1(\mathbf{u}) \, \mathrm{L}_1(u_1, u_2) \, \mathrm{L}_2(v_1, v_2) = \mathrm{L}_1(v_1, u_2) \, \mathrm{L}_2(u_1, v_2) \, \mathrm{R}_1(\mathbf{u})$$

$$\mathrm{R}_2(\mathbf{u}) \, \mathrm{L}_1(u_1, u_2) \, \mathrm{L}_2(v_1, v_2) = \mathrm{L}_1(u_1, v_2) \, \mathrm{L}_2(v_1, u_2) \, \mathrm{R}_2(\mathbf{u})$$



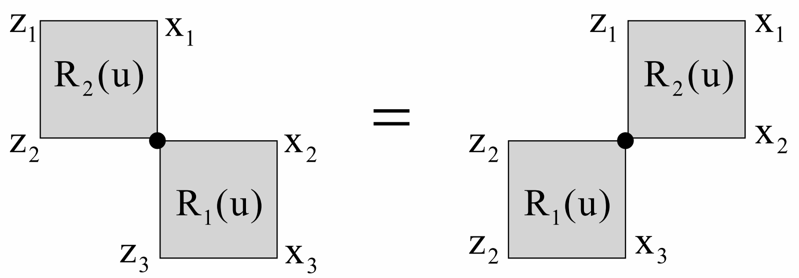
$$\mathbf{r}_1 \mathbf{r}_2 = \mathbf{r}_2 \mathbf{r}_1$$

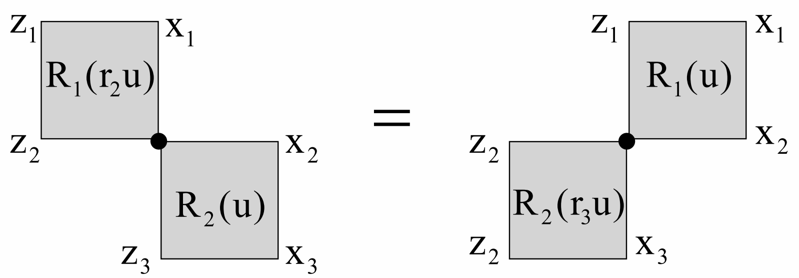
Yang-Baxter equation

relations in the more complicated case: $\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2)\mathrm{L}_3(w_1,w_2)$

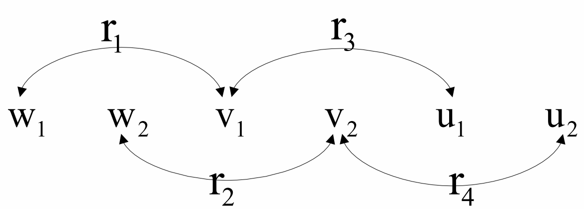
$$r_4 r_1 = r_1 r_4 \leftrightarrow \mathrm{R}^2_{12}(r_1\mathbf{u})\mathrm{R}^1_{23}(\mathbf{u}) = \mathrm{R}^1_{23}(r_4\mathbf{u})\mathrm{R}^2_{12}(\mathbf{u})$$

$$r_2 r_3 = r_3 r_2 \leftrightarrow \mathrm{R}^2_{23}(r_3\mathbf{u})\mathrm{R}^1_{12}(\mathbf{u}) = \mathrm{R}^1_{12}(r_2\mathbf{u})\mathrm{R}^2_{23}(\mathbf{u})$$





$$\mathrm{u} = (\mathrm{w}_1\ \mathrm{w}_2\ \mathrm{v}_1\ \mathrm{v}_2\ \mathrm{u}_1\ \mathrm{u}_2)$$



Factorization of general transfer matrix

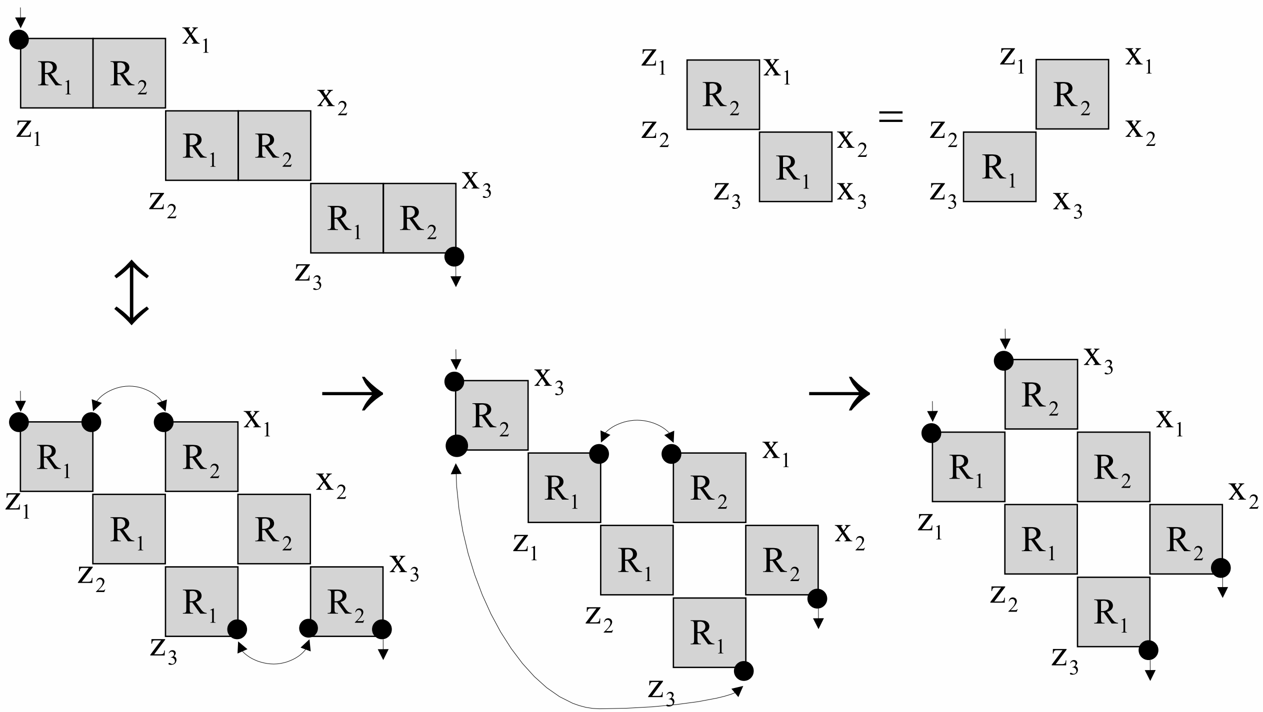
$$\mathbb{T}_{\boldsymbol{\rho}}(u) = \mathrm{tr}_{\mathbb{V}_0}\, \mathrm{R}_{10}(u)\mathrm{R}_{20}(u)\ldots\mathrm{R}_{N0}(u)\ ;\ \boldsymbol{\rho} = (\rho_1, \rho_2)$$

commutativity: $$\mathbb{T}_{\boldsymbol{\rho}}(u)\mathbb{T}_{\boldsymbol{\rho}'}(v) = \mathbb{T}_{\boldsymbol{\rho}'}(v)\mathbb{T}_{\boldsymbol{\rho}}(u)$$

factorization: $$\mathbb{T}_{\boldsymbol{\rho}}(u) = \mathrm{Q}_1(u+\rho_1)\cdot\mathrm{P}\cdot\mathrm{Q}_2(u+\rho_2)\ \ ;\ \ \mathrm{P}\Phi(z_1,\cdots,z_N) = \Phi(z_N, z_1,\cdots z_{N-1})$$

$$\mathrm{Q}_2(u) = \mathrm{tr}_{\mathbb{V}_0}\, \mathrm{P}_{10}\mathrm{R}^2_{10}(u)\cdot\mathrm{P}_{20}\mathrm{R}^2_{20}(u)\cdots\mathrm{P}_{N0}\mathrm{R}^2_{N0}(u),$$

$$\mathrm{Q}_1(u) = \mathrm{tr}_{\mathbb{V}_0}\, \mathrm{P}_{10}\mathrm{R}^1_{10}(u)\cdot\mathrm{P}_{20}\mathrm{R}^1_{20}(u)\cdots\mathrm{P}_{N0}\mathrm{R}^1_{N0}(u)$$



connection between Q-operators and degeneration points

connection between Q-operators

$$\mathrm{Q}_1(u) = \mathrm{Q}_1(u_1, u_2) \quad ; \quad \mathrm{Q}_2(v) = \mathrm{Q}_2(v_1, v_2)$$

$$\mathrm{Q}_2(u_1, u_2) \cdot \mathrm{S} = \mathrm{S} \cdot \mathrm{Q}_1(u_2, u_1) \;\; ; \;\; \mathrm{S} = [i\partial_1]^{u_1 - u_2} \cdots [i\partial_N]^{u_1 - u_2}$$

degeneration points for R-operators

$$\mathrm{R}_1(\mathbf{u})|_{v_1 = u_1} = \mathbb{1} \, ; \;\; \mathrm{R}_2(\mathbf{u})|_{v_2 = u_2} = \mathbb{1}$$

$$\mathrm{R}_k(u) = \left. \mathrm{R}_k(\mathbf{u})\right|_{v=0} \to \quad \mathrm{R}_k(u)|_{u = \sigma_k} = \mathbb{1}$$

degeneration points for Q-operators

$$\mathrm{Q}_1(\sigma_1) = \mathrm{P}^{-1} \, ; \; \mathrm{Q}_2(\sigma_2) = \mathrm{P}^{-1}$$

degeneration points for transfer matrix

$$\mathbb{T}_{\boldsymbol{\rho}}(u) = \mathrm{Q}_1(u + \rho_1) \cdot \mathrm{P} \cdot \mathrm{Q}_2(u + \rho_2)$$

$$\boldsymbol{\rho}_2 = (\sigma_1 - u, \sigma_2 + u) \, ; \; \boldsymbol{\rho}_1 = (\sigma_1 + u, \sigma_2 - u)$$

$$\mathbb{T}_{\boldsymbol{\rho}_2}(u) = \mathrm{Q}_2(2u + \sigma_2) \, ; \; \mathbb{T}_{\boldsymbol{\rho}_1}(u) = \mathrm{Q}_1(2u + \sigma_1)$$

commutativity

$$[\mathrm{Q}_1(u), \mathrm{Q}_1(v)] = [\mathrm{Q}_1(u), \mathrm{Q}_2(v)] = [\mathrm{Q}_2(u), \mathrm{Q}_2(v)] = 0$$

Baxter equation

defining equation for the operator $\mathrm{R}_2(\mathbf{u})$

$$\mathrm{R}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2)=\mathrm{L}_1(u_1,v_2)\,\mathrm{L}_2(v_1,u_2)\,\mathrm{R}_2(\mathbf{u})\ \ ;\ \ [\mathrm{R}_2(\mathbf{u}),z_2]=0$$

$$\mathrm{L}_1(u)=\mathrm{Z}_1\cdot\begin{pmatrix}u_1 & -\partial_{z_1}\\ 0 & u_2\end{pmatrix}\cdot\mathrm{Z}_1^{-1}\ ;\ \mathrm{L}_2(v)=\mathrm{Z}_2\cdot\begin{pmatrix}v_1 & -\partial_{z_2}\\ 0 & v_2\end{pmatrix}\cdot\mathrm{Z}_2^{-1}\ ;\ \mathrm{Z}_k=\begin{pmatrix}1 & 0\\ z_k & 1\end{pmatrix}$$

$$\mathrm{Z}_1^{-1}\,\mathrm{R}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{Z}_2=\begin{pmatrix}u_1 & -\partial_{z_1}\\ 0 & v_2\end{pmatrix}\begin{pmatrix}1 & 0\\ -z_{12} & 1\end{pmatrix}\begin{pmatrix}v_1 & -\partial_{z_2}\\ 0 & u_2\end{pmatrix}\cdot\mathrm{R}_2(\mathbf{u})\cdot\begin{pmatrix}v_1 & -\partial_{z_2}\\ 0 & v_2\end{pmatrix}^{-1}$$

$$\mathrm{Z}_1^{-1}\,\mathrm{R}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{Z}_2=\begin{pmatrix}\mathrm{R}_2(\mathbf{u})|_{u_k\to u_k+1}+v_2\mathrm{R}_2(\mathbf{u}) & -\mathrm{R}_2(\mathbf{u})\partial_{z_1}\\ -v_2\,z_{12}\,\mathrm{R}_2(\mathbf{u}) & (u_1-v_2)(u_2-v_2)\cdot\ \mathrm{R}_2(\mathbf{u})|_{u_k\to u_k-1}\end{pmatrix}$$

Baxter equation

$$\mathrm{P}_{k0}\mathrm{R}^2_{k0}(u)\,\mathrm{L}_k(u_1,u_2)=\mathrm{Z}_0\cdot\begin{pmatrix}\mathrm{P}_{k0}\mathrm{R}^2_{k0}(u+1) & -\mathrm{P}_{k0}\mathrm{R}^2_{k0}(u))\partial_k\\ 0 & u_1\,u_2\,\mathrm{P}_{k0}\mathrm{R}^2_{k0}(u-1)\end{pmatrix}\cdot\mathrm{Z}_0^{-1}$$

$$\mathrm{Q}_2(u)=\mathrm{tr}_{\mathbb{V}_0}\,\mathrm{P}_{10}\mathrm{R}^2_{10}(u)\cdots\mathrm{P}_{N0}\mathrm{R}^2_{N0}(u)\quad;\quad \mathrm{t}(u)=\mathrm{tr}\,\mathrm{L}_1(u)\mathrm{L}_2(u)\cdots\mathrm{L}_N(u)$$

$$\mathrm{t}(u)\mathrm{Q}_2(u)=\mathrm{Q}_2(u+1)+(u_1u_2)^N\cdot\mathrm{Q}_2(u-1)$$

iterative diagonalization of $\mathrm{B}(u)$

$$\mathrm{S}_-\mathrm{U}(\mathbf{z},\mathbf{x},p) = p\cdot\mathrm{U}(\mathbf{z},\mathbf{x},p)\ ;\ \mathrm{B}(u)\mathrm{U}(\mathbf{z},\mathbf{x},p) = p\cdot(u-x_1)(u-x_2)\cdots(u-x_{N-1})\mathrm{U}(\mathbf{z},\mathbf{x},p)$$

defining property of operator R_2

$$\Lambda_N(\mathbf{u}) = \mathrm{R}^2_{12}(\mathbf{u})\mathrm{R}^2_{23}(\mathbf{u})\cdots\mathrm{R}^2_{N-1,N}(\mathbf{u})$$

$$\Lambda_N(\mathbf{u})\cdot\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(u_1,u_2)\cdots\mathrm{L}_N(u_1,v_2) = \mathrm{L}_1(u_1,v_2)\,\mathrm{L}_2(u_1,u_2)\cdots\mathrm{L}_N(u_1,u_2)\cdot\Lambda_N(\mathbf{u})$$

applying to function $\Psi(z_1\cdots z_{N-1})$ which does not depend on z_N

$$\mathrm{L}_N(u_1,v_2)\to\begin{pmatrix} u_1 & 0\\ (u_1-v_2+1)\cdot z_N & v_2\end{pmatrix}\ ;\ \mathrm{L}_1(u_1,v_2) = \begin{pmatrix} 1 & 0\\ z_1\cdot\frac{u_2-v_2}{u_2} & \frac{v_2}{u_2}\end{pmatrix}\cdot\mathrm{L}_1(u_1,u_2)$$

$$\Lambda_N(\mathbf{u})\cdot\begin{pmatrix}\mathrm{A}_{N-1}(u) & \mathrm{B}_{N-1}(u)\\ \mathrm{C}_{N-1}(u) & \mathrm{D}_{N-1}(u)\end{pmatrix}\cdot\Psi(z_1\cdots z_{N-1})\cdot\begin{pmatrix} u_1 & 0\\ (u_1-v_2+1)\cdot z_N & v_2\end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0\\ z_1\cdot\frac{u_2-v_2}{u_2} & \frac{v_2}{u_2}\end{pmatrix}\begin{pmatrix}\mathrm{A}_N(u) & \mathrm{B}_N(u)\\ \mathrm{C}_N(u) & \mathrm{D}_N(u)\end{pmatrix}\cdot\Lambda_N(\mathbf{u})\,\Psi(z_1\cdots z_{N-1})$$

relation (12)

$$\mathrm{B}_N(u)\cdot\Lambda_N(\mathbf{u})\,\Psi(z_1\cdots z_{N-1}) = v_2\cdot\Lambda_N(\mathbf{u})\cdot\mathrm{B}_{N-1}(u)\Psi(z_1\cdots z_{N-1})$$

iterative diagonalization of $\mathrm{B}(u)$

usual notations with spectral parameter

$$x = v_2 - u \quad ; \quad \Lambda_N(x) = \mathrm{R}^2_{12}(x)\mathrm{R}^2_{23}(x)\cdots\mathrm{R}^2_{N-1,N}(x)$$

$$\mathrm{B}_N(u)\cdot\Lambda_N(x)\Psi(z_1\cdots z_{N-1}) = (u-x)\cdot\Lambda_N\,(x)\cdot\mathrm{B}_{N-1}(u)\Psi(z_1\cdots z_{N-1})$$

iteration

$$\mathrm{B}_N(u)\cdot\Lambda_N\,(x_1)\cdots\Lambda_2(x_{N-1})\Psi(z_1) = (u-x_1)\cdot(u-x_{N-1})\cdot\mathrm{B}_1(u)\cdot\Psi(z_1)$$

$$\Lambda_k(x) = \mathrm{R}^2_{12}(x)\mathrm{R}^2_{23}(x)\cdots\mathrm{R}^2_{k-1,k}(x)$$

one-point operator

$$\mathrm{B}_1(u) = -\partial_1 \;\; ; \;\; \mathrm{B}_1(u)\mathrm{e}^{-pz_1} = p\cdot\mathrm{e}^{-pz_1}$$

answer

$$\mathrm{B}(u)\mathrm{U}(\mathbf{z},\mathbf{x},p) = p\cdot(u-x_1)(u-x_2)\cdots(u-x_{N-1})\mathrm{U}(\mathbf{z},\mathbf{x},p)$$

$$\mathrm{U}(\mathbf{z},\mathbf{x},p) = \Lambda_N\,(x_1)\cdots\Lambda_2(x_{N-1})\mathrm{e}^{-pz_1}$$

connection with Q_2-operator

$$\mathrm{Q}_2(x) = \mathrm{R}^2_{12}(x)\mathrm{R}^2_{23}(x)\cdots\mathrm{R}^2_{N-1,N}(x)\mathrm{R}^2_{N,0}(x)\Big|_{z_0=z_1} \;\; ; \; \mathrm{U}(\mathbf{z},\mathbf{x},p) = \mathrm{Q}_2\,(x_1)\cdots\mathrm{Q}_2\,(x_{N-1})\,\mathrm{e}^{-pz_1}$$

representations of $GL(n, C)$

$$\mathrm{z} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{21} & 1 & 0 & \dots & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & z_{n3} & \dots & 1 \end{pmatrix} \;\; ; \;\; \mathrm{h} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1n} \\ 0 & h_{22} & h_{23} & \dots & h_{2n} \\ 0 & 0 & h_{33} & \dots & h_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_{n,n} \end{pmatrix}$$

main construction: representation with labels $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ and $\bar{\boldsymbol{\sigma}} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_n)$, $\sigma_k - \bar{\sigma}_k \in \mathbb{Z}$

$$g^{-1} \cdot \mathrm{z} = \mathrm{z}' \cdot \mathrm{h} \quad ; \quad \mathrm{T}(g)\, \Phi(\mathrm{z}) = [h_{11}]^{\sigma_1 + 1} [h_{22}]^{\sigma_2 + 2} \cdots [h_{nn}]^{\sigma_n + n} \cdot \Phi(\mathrm{z}')$$

intertwining operators: representations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_k = \mathrm{s}_k \boldsymbol{\sigma}$ are equivalent

$$\mathrm{S}_k \, \mathrm{T}^{\boldsymbol{\sigma}} \;=\; \mathrm{T}^{\boldsymbol{\sigma}_k} \mathrm{S}_k \; ; \;\; k = 1, \cdots, n-1$$

$$\mathrm{s}_k \, (\dots \sigma_k, \sigma_{k+1}, \dots) = (\dots \sigma_{k+1}, \sigma_k, \dots)$$

$$\mathrm{S}_k = [i\mathrm{D}_k]^{\sigma_{k,k+1}} \;\; ; \;\; \mathrm{D}_k = \frac{\partial}{\partial z_{k+1,k}} + \sum_{m=k+2}^{n} z_{m,k+1} \, \frac{\partial}{\partial z_{mk}}$$

representations of $GL(n,C)$

L-operator: $\mathrm{L}(u) = u + \mathbf{e}_{ik}\mathrm{E}_{ki}$

explicit formulae for $GL(3,C)$

$$\mathrm{L}(u) = \begin{pmatrix} u+\mathrm{E}_{11} & \mathrm{E}_{21} & \mathrm{E}_{31} \\ \mathrm{E}_{12} & u+\mathrm{E}_{22} & \mathrm{E}_{32} \\ \mathrm{E}_{13} & \mathrm{E}_{23} & u+\mathrm{E}_{33} \end{pmatrix}$$

$$\mathrm{L}(u) = \begin{pmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{pmatrix} \begin{pmatrix} u-\sigma_1 & -\mathrm{D}_1 & -\partial_{z_{31}} \\ 0 & u-\sigma_2 & -\mathrm{D}_2 \\ 0 & 0 & u-\sigma_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{pmatrix}^{-1}$$

$$\mathrm{D}_1 = \partial_{z_{21}} + z_{32}\partial_{z_{31}} \quad ; \quad \mathrm{D}_2 = \partial_{z_{32}}$$

general structure

diagonal: $u-\sigma_1, u-\sigma_2, \cdots, u-\sigma_n \leftrightarrow u_1, u_2, \cdots, u_n$

next diagonal: $-\mathrm{D}_1, -\mathrm{D}_2, \cdots, -\mathrm{D}_{n-1}$

intertwining operators S_k exchange parameters in L-operator

$$\mathrm{S}_k = [i\mathrm{D}_k]^{u_{k+1}-u_k} \quad ; \quad \mathrm{S}_k\,\mathrm{L}(u_1,\ldots,u_k,u_{k+1},\cdots u_n) = \mathrm{L}(u_1,\ldots,u_{k+1},u_k,\cdots u_n)\,\mathrm{S}_k$$

Yang-Baxter equation

defining relation

$$\check{\mathrm{R}}(u-v)\,\mathrm{L}_1(u_1\cdots u_n)\,\mathrm{L}_2(v_1\cdots v_n) = \mathrm{L}_1(v_1\cdots v_n)\,\mathrm{L}_2(u_1\cdots u_n)\,\check{\mathrm{R}}(u-v)$$

permutation group $\mathfrak{S}_{2n}$ of $2n$ parameters: $\mathbf{u} \equiv (v_1,\cdots,v_n,u_1,\cdots,u_n)$

$$s \leftrightarrow \check{\mathrm{R}} \quad ; \quad (v_1,\dots,v_n,u_1,\dots,u_n) \xrightarrow{s} (u_1,\dots,u_n,v_1,\dots,v_n)$$

$$(\overbrace{v_1\ ,\ \dots\ ,\ v_n}^{\mathbb{S}_1\ ,\ \cdots\ ,\ \mathbb{S}_{n-1}},\ \overbrace{u_1\ ,\ \dots\ ,\ u_n}^{\mathbb{S}_{n+1}\ ,\ \cdots\ ,\ \mathbb{S}_{2n-1}}\ )\ ;\ \mathbb{S}_k = \begin{cases} \mathbb{1}\otimes \mathrm{S}_k\,, & k=1,\dots,n-1 \\ \mathrm{S}_{k-n}\otimes\mathbb{1}\,, & k=n+1,\dots,2n-1 \end{cases}$$

$$\mathbb{S}_k\,\mathrm{L}_2(v_1,\dots,v_k,v_{k+1},\cdots v_n) = \mathrm{L}_2(v_1,\dots,v_k,v_{k+1},\cdots v_n)\,\mathbb{S}_k\ ;\ k=1\cdots n-1$$

$$\mathbb{S}_{n-k}\,\mathrm{L}_1(u_1,\dots,u_k,u_{k+1},\cdots u_n) = \mathrm{L}_1(u_1,\dots,u_{k+1},u_k,\cdots u_n)\,\mathbb{S}_{n-k}\ ;\ k=n+1\cdots 2n-1$$

$$(v_1,\dots,\overbrace{v_n,u_1}^{\mathbb{S}_n},\dots,u_n).$$

$$\mathbb{S}_n\,\mathrm{L}_1(u_1,u_2,\dots,u_n)\,\mathrm{L}_2(v_1,\dots,v_{n-1},v_n) = \mathrm{L}_1(v_n,u_2,\dots,u_n)\,\mathrm{L}_2(v_1,\dots,v_{n-1},u_1)\,\mathbb{S}_n\,.$$

$$\mathbb{S}_n = \left[\mathrm{z_2}^{-1}\mathrm{z}_1\right]^{u_1-v_n}_{n1} \quad ; \quad \mathrm{z_2}^{-1}\mathrm{z}_1 = \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_1-z_2 & 1 \end{pmatrix}$$

Yang-Baxter equation

defining relations for generators

$$\mathbb{S}_k(s_k\mathbf{u})\mathbb{S}_k(\mathbf{u}) = \mathbb{1}$$

$$\mathbb{S}_i(s_k\mathbf{u})\mathbb{S}_k(\mathbf{u}) = \mathbb{S}_k(s_i\mathbf{u})\mathbb{S}_i(\mathbf{u})\ ;\ |i-k|>1$$

$$\mathbb{S}_k(s_{k+1}s_k\mathbf{u})\mathbb{S}_{k+1}(s_k\mathbf{u})\mathbb{S}_k(\mathbf{u}) = \mathbb{S}_{k+1}(s_k s_{k+1}\mathbf{u})\mathbb{S}_k(s_{k+1}\mathbf{u})\mathbb{S}_{k+1}(\mathbf{u})$$

useful decomposition on permutations $u_k \leftrightarrow v_k$: $\mathfrak{S}_{2n} \to \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_2$

$$(u_1,\ldots,u_k,\ldots,v_k,\ldots,v_n) \xrightarrow{\mathrm{r}_k} (u_1,\ldots,v_k,\ldots,u_k,\ldots,v_n)$$

$$\mathrm{R}_k\,\mathrm{L}_1(u_1\ldots u_k\ldots u_n)\,\mathrm{L}_2(v_1\ldots v_k\ldots v_n) = \mathrm{L}_1(u_1\ldots v_k\ldots u_n)\,\mathrm{L}_2(v_1\ldots u_k\ldots v_n)\,\mathrm{R}_k$$

$$\mathrm{R}_k = (\mathbb{S}_{n+k-1}\ldots\mathbb{S}_{n+1})\,(\mathbb{S}_k\ldots\mathbb{S}_{n-1})\,\mathbb{S}_n\,(\mathbb{S}_{n-1}\ldots\mathbb{S}_k)\,(\mathbb{S}_{n+1}\ldots\mathbb{S}_{n+k-1})$$

factorization of R-operator and relations needed for factorization of transfer-matrix

$$\check{\mathrm{R}}(\mathbf{u}) = \mathrm{R}_1(r_2\cdots r_n\mathbf{u})\ \ldots\mathrm{R}_{n-1}(r_n\mathbf{u})\mathrm{R}_n(\mathbf{u})$$

$$\mathrm{R}^k_{12}(\mathbf{u})\,\mathrm{R}^i_{23}(\mathbf{u}) = \mathrm{R}^i_{23}(\mathbf{u})\mathrm{R}^k_{12}(\mathbf{u})\quad ;\ k>i$$

Factorization of transfer matrix and degeneration points

factorization of transfer-matrix

$$\mathbb{T}_{\boldsymbol{\rho}}(u) = \mathrm{tr}_{\mathbb{V}_0}\,\mathrm{R}_{10}(u)\mathrm{R}_{20}(u)\ldots\mathrm{R}_{N0}(u)\ ;\ \boldsymbol{\rho} = (\rho_1,\cdots,\rho_n)$$

$$\mathbb{T}_{\boldsymbol{\rho}}(u) = \mathrm{Q}_1(u+\rho_1)\cdot\mathrm{P}\cdot\mathrm{Q}_2(u+\rho_2)\cdots\mathrm{P}\cdot\mathrm{Q}_n(u+\rho_n)$$

$$\mathrm{Q}_k(u) = \mathrm{tr}_{\mathbb{V}_0}\,\mathrm{P}_{10}\mathrm{R}^k_{10}(u)\cdot\mathrm{P}_{20}\mathrm{R}^k_{20}(u)\cdots\mathrm{P}_{N0}\mathrm{R}^k_{N0}(u)$$

degeneration points for R-operators

$$\mathrm{R}_k(\mathbf{u})\big|_{v_k=u_k} = \mathbb{1}$$

$$\mathrm{R}_k(u) = \mathrm{R}_k(\mathbf{u})\big|_{v=0} \rightarrow \quad \mathrm{R}_k(u)\big|_{u=\sigma_k} = \mathbb{1}$$

degeneration points for Q-operators

$$\mathrm{Q}_k(\sigma_k) = \mathrm{P}^{-1}$$

degeneration points for transfer matrix

$$\boldsymbol{\rho}_k = (\sigma_1 - u, \sigma_2 - u, \cdots, \sigma_k + u\cdot(n-1), \cdots, \sigma_n - u)$$

$$\mathbb{T}_{\boldsymbol{\rho}_k}(u) = \mathrm{Q}_k\left(nu + \sigma_k\right)$$

commutativity

$$\mathbb{T}_{\boldsymbol{\rho}}(u)\mathbb{T}_{\boldsymbol{\rho}'}(v) = \mathbb{T}_{\boldsymbol{\rho}'}(v)\mathbb{T}_{\boldsymbol{\rho}}(u) \quad\rightarrow\quad [\mathrm{Q}_i(u), \mathrm{Q}_k(v)] = 0$$

Baxter equation

defining relation for R_n

$$\mathrm{R}^n_{k0}(\mathbf{u})\,\mathrm{L}_k(u_1,\ldots,u_n)\,\mathrm{L}_0(v_1,\ldots,v_n) = \mathrm{L}_k(u_1,\ldots,v_n)\,\mathrm{L}_0(v_1,\ldots,u_n)\,\mathrm{R}^n_{k0}(\mathbf{u})$$

at the point $v_n = 0$ can be rewritten in the form

$$\mathrm{P}_{k0}\mathrm{R}^n_{k0}(u)\,\mathrm{L}_k(u_1,\ldots,u_n) = \mathrm{Z}_0\cdot\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & u_1\cdots u_n\cdot\mathrm{P}_{k0}\mathrm{R}^n_{k0}(u-1) \end{pmatrix}\cdot\mathrm{Z}_0^{-1}$$

transfer matrices

$$\mathbb{T}(u) = \mathrm{L}_1(u)\cdots\mathrm{L}_N(u)$$

$$\mathrm{t}_k(u) = \mathrm{tr}_k\,\mathbb{A}_k\,\mathbb{T}_1(u)\mathbb{T}_2(u+1)\cdots\mathbb{T}_k(u+k-1)$$

$\mathbb{A}_k$ - anti-symmetrization of k indices and trace in $\otimes^k\mathbb{C}^n$
explicit formulae

$$(\mathbb{A}_k)^{b_1b_2\ldots b_k}_{a_1a_2\ldots a_k} = \frac{1}{k!}\sum_p(-)^{sign(p)}\delta^{b_1}_{p(a_1)}\delta^{b_2}_{p(a_2)}\cdots\delta^{b_k}_{p(a_k)}$$

$$\mathrm{t}_k(u) = \sum_{a_1<a_2<\ldots<a_k}\sum_p(-)^{sign(p)}\,\mathbb{T}_{p(a_1)a_1}(u)\mathbb{T}_{p(a_2)a_2}(u+1)\cdots\mathbb{T}_{p(a_k)a_k}(u+k-1)$$

Baxter equation

examples of Baxter equation

$$\mathrm{A}(u) = (u_1 \cdots u_n)^N \ ; \ \mathrm{t}_n(u) = \mathrm{A}(u+n-1) \ ; \ \mathrm{Q}(u) = \mathrm{Q}_n(u)$$

✔ $n = 2$

$$\mathrm{A}(u+1)\cdot \mathrm{Q}(u)\mathrm{t}_1(u) - \mathrm{Q}(u+1)\mathrm{t}_2(u) = \mathrm{A}(u)\mathrm{A}(u+1)\cdot \mathrm{Q}(u-1)$$

$$\mathrm{Q}(u)\mathrm{t}_1(u) - \mathrm{Q}(u+1) = u_1^N\, u_2^N \cdot \mathrm{Q}(u-1)$$

✔ $n = 3$

$$\mathrm{A}(u{+}1)\mathrm{A}(u{+}2)\cdot \mathrm{Q}(u)\mathrm{t}_1(u) - \mathrm{A}(u{+}2)\cdot \mathrm{Q}(u{+}1)\mathrm{t}_2(u) + \mathrm{Q}(u{+}2)\mathrm{t}_3(u) = \mathrm{A}(u)\mathrm{A}(u{+}1)\mathrm{A}(u{+}2)\cdot \mathrm{Q}(u{-}1)$$

$$\mathrm{A}(u+1)\cdot \mathrm{Q}(u)\mathrm{t}_1(u) - \mathrm{Q}(u+1)\mathrm{t}_2(u) + \mathrm{Q}(u+2) = \mathrm{A}(u)\mathrm{A}(u+1)\cdot \mathrm{Q}(u-1)$$

in the general case all can be packed in one formula

$$\mathrm{tr}_n\, \mathbb{A}_n \left[\mathrm{e}^{\partial_u}\, \mathbb{T}_1(u) - \mathrm{A}(u+n)\right] \left[\mathrm{e}^{\partial_u}\, \mathbb{T}_2(u) - \mathrm{A}(u+n-1)\right] \left[\mathrm{e}^{\partial_u}\, \mathbb{T}_n(u) - \mathrm{A}(u+1)\right] \cdot Q_n(u) = 0$$

$\mathbb{A}_n$ - anti-symmetrization of n indices and trace in $\otimes^n \mathbb{C}^n$ [conjectured by Chervov-Talalaev]

fusion relations

- For integer points: $\rho_k - \rho_{k+1} \in \mathbb{N}$ the representation with label $\boldsymbol{\rho}$ is not irreducible: in infinite-dimensional space $\mathbb{V}_{\boldsymbol{\rho}}$ there exists finite-dimensional invariant subspace.
- Transfer-matrix with this finite-dimensional auxiliary space $\mathrm{t}_{\boldsymbol{\rho}}(u)$ can be expressed in terms of $\mathbb{T}_{\boldsymbol{\rho}}(u)$

$$\mathrm{t}_{\boldsymbol{\rho}}(u) = \sum_{\mathrm{p}} (-)^{\mathrm{sign}(\mathrm{p})} \cdot \mathbb{T}_{\mathrm{p}\boldsymbol{\rho}}(u) \tag{1}$$

  where $\mathrm{p}\boldsymbol{\rho} = (\rho_{k_1}, \rho_{k_2} \cdots \rho_{k_n})$ is permutation of $(\rho_1, \rho_2 \cdots \rho_n)$. The sum is over all permutations.
- The factorization of $\mathbb{T}_{\boldsymbol{\rho}}(u)$ into product of Q-operators leads to the following determinant representation for $\mathrm{t}_{\boldsymbol{\rho}}(u)$

$$\mathrm{t}_{\boldsymbol{\rho}}(u) = \mathrm{P}^{n-1} \cdot \begin{vmatrix} \mathrm{Q}_1(u+\rho_1) & \mathrm{Q}_2(u+\rho_1) & \cdots & \mathrm{Q}_n(u+\rho_1) \\ \mathrm{Q}_1(u+\rho_2) & \mathrm{Q}_2(u+\rho_2) & \cdots & \mathrm{Q}_n(u+\rho_2) \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Q}_1(u+\rho_n) & \mathrm{Q}_2(u+\rho_n) & \cdots & \mathrm{Q}_n(u+\rho_n) \end{vmatrix}$$

conclusions

- simplest building block – operator $\mathrm{R}_n(\mathbf{u})$ which interchanges $u_n \leftrightarrow v_n$

$$\mathrm{R}_n(\mathbf{u})\,\mathrm{L}_1(u_1,\dots,u_n)\,\mathrm{L}_2(v_1,\dots,v_n) = \mathrm{L}_1(u_1,\dots,v_n)\,\mathrm{L}_2(v_1,\dots,u_n)\,\mathrm{R}_n(\mathbf{u})$$

- $\mathrm{R}_n(\mathbf{u}) \;\leftrightarrow\; Q$-operator $\;\leftrightarrow\;$ $\mathrm{U}(\mathbf{z},\mathbf{x})$ - transition operator to the SOV-representation

$$\Psi(\mathbf{z}) = \int \mathrm{U}(\mathbf{z},\mathbf{x})\,\mathrm{Q}(\mathbf{x})\,\mu(\mathbf{x})\,\mathrm{d}^N\mathbf{x}$$

- The last step is still missing for $SL(n,C)$.