Multiloop Calculation of Renormalization Constants: R-operation as a technique for representation of residues in the poles of graphs in the form of convergent integrals

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2010-10-20

Conformal bootstrap in sigma-model, 3rd order of 1/n expansion.

- A.N. Vasil'ev, Yu.M. Pis'mak, and Yu.R. Khonkonen. Theor. Math. Phys., V.50, N2, P.127 (1982)

Model A of critical dynamics, 3rd order of ϵ-expansion.

- N.V. Antonov, A.N. Vasil'ev. Theor. Math. Phys., V.60, N1, P.671 (1984)

Uniqueness method.

- A.N. Vasil'ev, Yu.M. Pis'mak, and Yu.R. Khonkonen. Theor. Math. Phys., V.47, N3, P.465 (1981)
- D.I. Kazakov. . Theor. Math. Phys., V.62, N1, P.84 (1985).

Model φ^4, 5-th order of ϵ-expansion:

- Chetyrkin K.G, Kataev A.L., Tkachev F.V., Phys.Lett., B99, 147 (1981); B101,457(E) (1981)
- Gorihny S.S., Larin S.A.,Tkachev F.V., Phys.Lett., A101, 120 (1984)
- Kazakov D.I., Phys.Lett., B133, 406 (1983)
- Kleinert H., Neu J., Shulte-Frohlinde V., Chetyrkin K.G., Larin S.A., Phys.Lett., B272,39 (1991); Erratum: B319, 545 (1993)

Theory of developed turbulence, 2nd order of ϵ-expansion.

- L.Ts. Adzhemyan, N.V. Antonov, M.V. Kompaniets and A.N. Vasil'ev, Int. J. Mod. Phys.B, V.17, P.2137, (2003).

Analytical calculations in the higher orders of perturbation theory of ϕ^4-model meet with significant difficulties

- H. Kleinert, V. Schulte-Frohlinde, Critical Properties of ϕ^4-Theories (2001).

Numerical calculations can be an alternative to the analytical ones. Precision of numerical calculations constantly increases with the increase of the computational power of computers.

In this report we present the easy and effective way for numerical calculation of graphs in any order of perturbation theory. The only limitation of the usage of this technique is the computational power of available computers.

The suggested method allows:

- Represent the pole residues of the graph as a convergent integrals
- Calculate graphs of any complexity due to universality and easiness of the method
- Perform a fully automated numerical calculation of renormalization constants

Outline

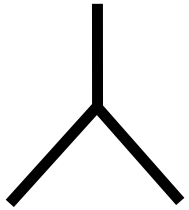
- Demonstration of our representation for renormalization constants, and example of its application to 4-loop graph
- Argumentation of the proposed representation
- Practical realization of R-operation suitable for numerical integration
- Examples of graph calculation
- 4-loop results for exponent η in ϕ^3 model

The φ^3 model in dimension $d = 6 - \varepsilon$

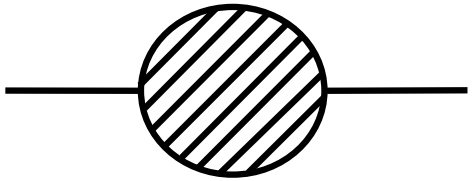
Renormalized action

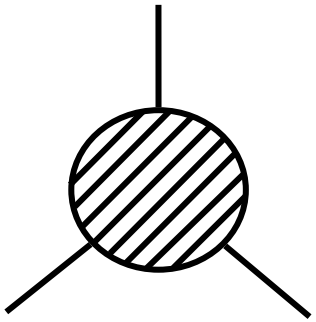
$$S = -\frac{1}{2}\left(\tau Z_1 + p^2 Z_2 + \Delta T_c\right)\varphi^2 - \frac{1}{6} g \mu^{\varepsilon/2} Z_3 \varphi^3$$

Line and vertex

 $= \dfrac{1}{k^2 + \tau}$, 

Superficially divergent Green functions are Γ_2 and Γ_3





We'll calculate renormalization constants on the basis of Bogolubov's R-operation

$$Z = 1 + KR'\Gamma$$

Here

K – operation that separates singular terms

R' – incomplete R-operation, which is related to R by $R = (1 - K)R'$, and corresponds to substraction of all divergent subgraphs of a given graph.

The proposed representation of renormalization constants

Let us consider a counter term in n-loop approximation and represent it as a sum of two terms

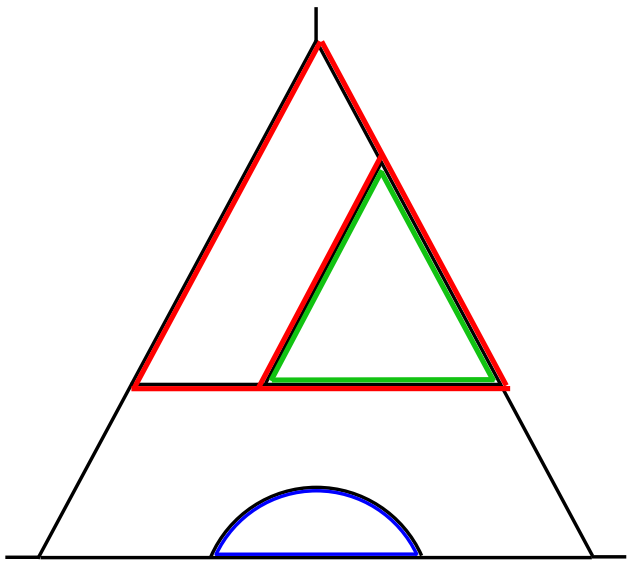
$$Z^{(n)} = KR'\Gamma^{(n)} = \frac{2}{n\epsilon}(\mathcal{N}\Gamma^{(n)} + \mathcal{J}\Gamma^{(n)}),$$

$\mathcal{N}\Gamma^{(n)}$ (determined as **genuine part**) is represented by an integral **finite** at $\epsilon = 0$, and contributes to the first-order pole only.

$\mathcal{J}\Gamma^{(n)}$ (**high-order pole part**) is constructed of low-order graphs of perturbation theory (the high-order poles are determined by the poles of low-order graphs)

Let us realize this representation on example

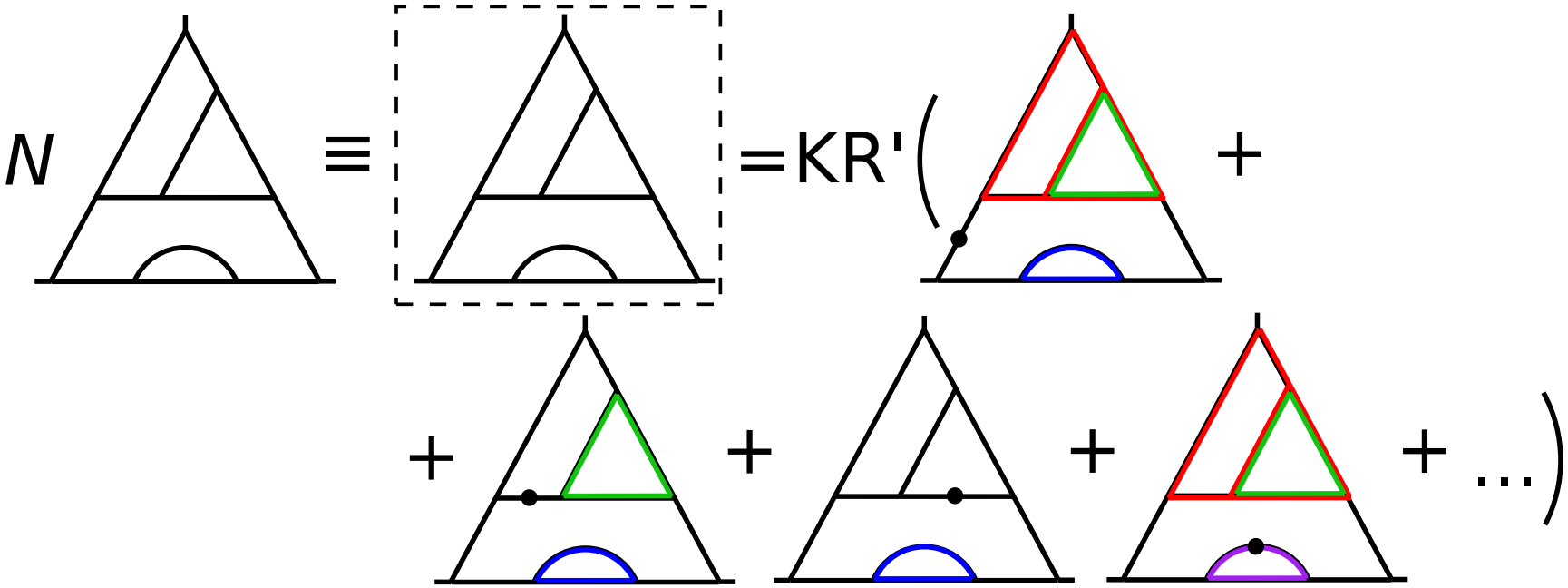
The example of the graph, which contributes to Z_3:



Relevant (divergent) subgraphs are marked with colored lines

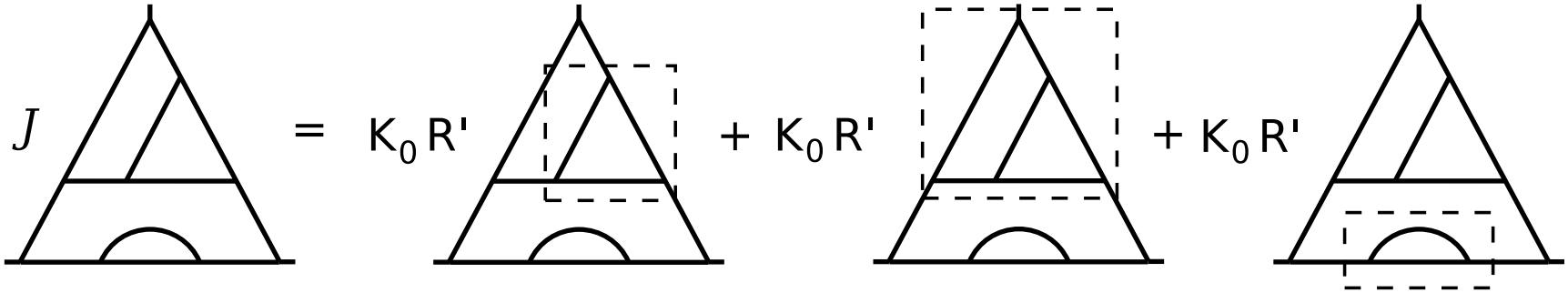
Genuine part $\mathcal{N}\gamma$ in representation

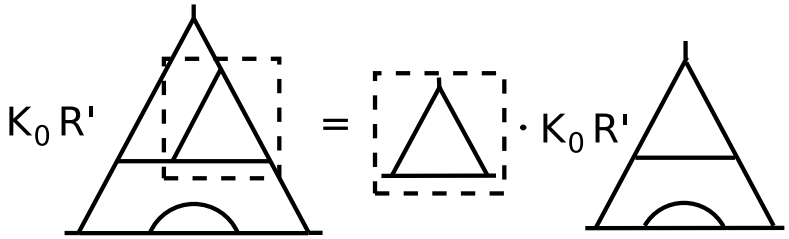
$$KR'\gamma = \frac{2}{n\epsilon}(\mathcal{N}\gamma + \mathcal{J}\gamma),$$

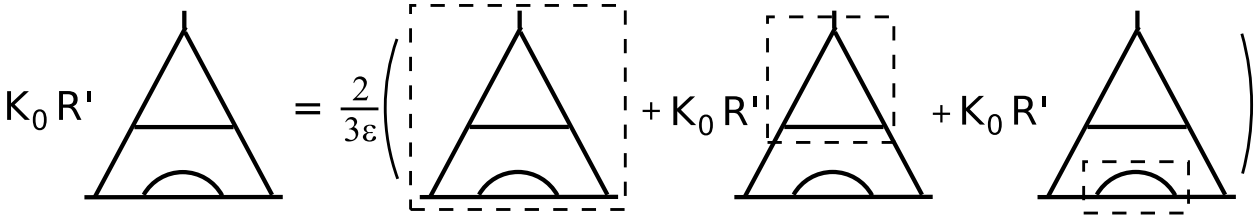


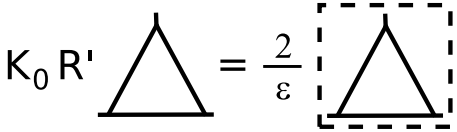
- Unity vertex is inserted into each line of the graph
- The above operation removes the superficial divergence
- KR' makes the expression finite at $\epsilon = 0$

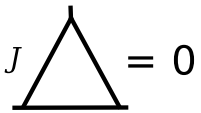
High-order pole part $\mathcal{J}\gamma$





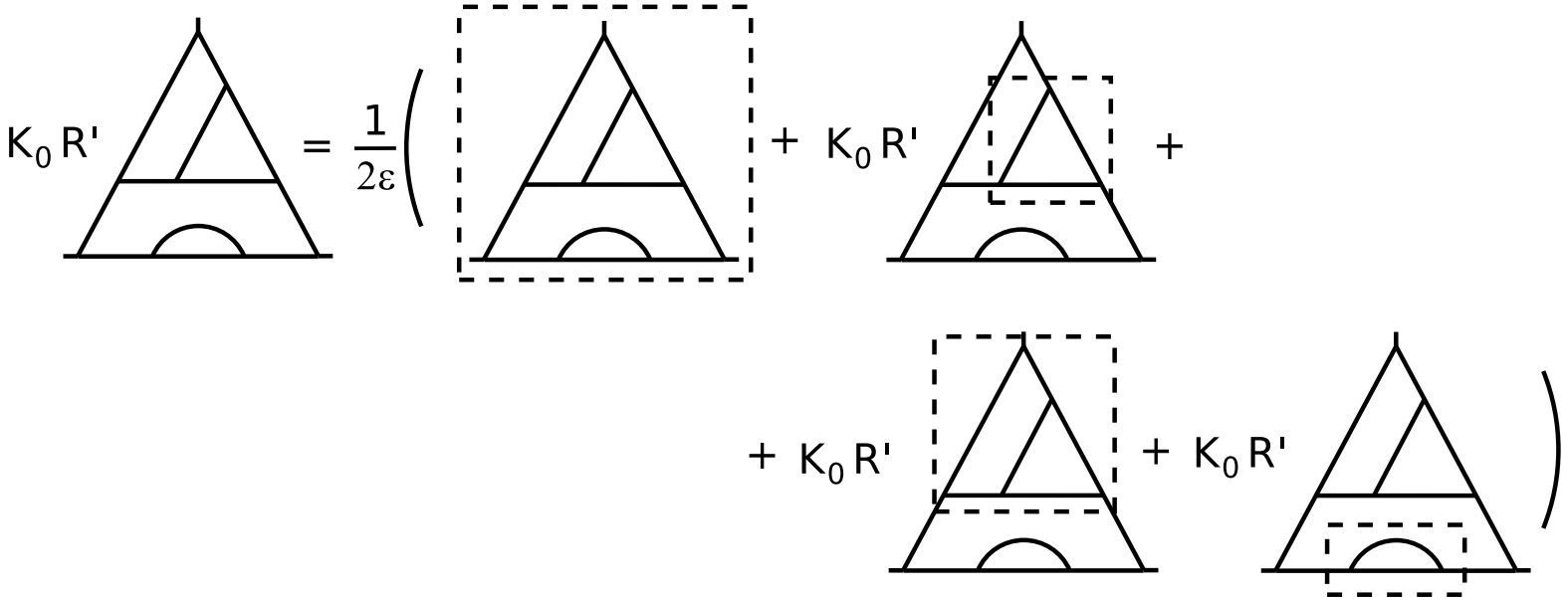






Resultant expression for counterterm

$$KR'\gamma = \frac{2}{n\epsilon}(\mathcal{N}\gamma + \mathcal{J}\gamma), \qquad n = 4$$



For calculation of counterterms it is sufficient to calculate genuine parts of graphs in every order of perturbation theory

Argumentation

Renormalization scheme

$$S = -\frac{1}{2}(\tau Z_1 + p^2 Z_2 + \Delta T_c)\varphi^2 - \frac{1}{6} g \mu^{\varepsilon/2} Z_3 \varphi^3$$

The following renormalization scheme is suitable for further calculations

- $\Gamma_2|_{p=0,\tau=0} = 0$
- $\Gamma_2|_{\otimes} = -\mu^2\,, \quad \{\otimes\} \equiv \{p = 0,\, \tau = \mu^2\}$ – normalization point
- $\frac{1}{2}\,\partial_p^2\,\Gamma_2|_{\otimes} = -1$
- $\Gamma_3|_{\otimes} = -g\mu^{\varepsilon/2}$

In this scheme renormalization constants depend on g and ϵ only.

K operation

$$K\Gamma_2 = \Gamma_2|_0 + \frac{\tau}{\mu^2}\left(\Gamma_2|_\otimes - \Gamma_2|_0\right) + \frac{p^2}{2}\left(\partial_p^2\Gamma_2|\right)_\otimes ,$$

$$K\Gamma_3 = \Gamma_3|_\otimes$$

Calculation of renormalization constants will be performed at $\tau = \mu^2$, then

$$K\Gamma_2 = (K_0 + p^2 K_2)\Gamma_2 , \qquad K\Gamma_3 = K_0\Gamma_3 ,$$

where

$$K_0 F(p) = F|_{p=0} , \qquad K_2 F(p) = \frac{1}{2}\partial_p^2 F|_{p=0} ,$$

K **operation separates the initial part of momentum series**.

Derivation of $Z^{(n)} = \frac{2}{n\varepsilon}\left(\mathcal{N}\Gamma^{(n)} + \mathcal{J}\Gamma^{(n)}\right)$ representation

Let Γ – be a logarithmic 1-irreducible Green function, which determines renormalization constant Z

$$Z = 1 + KR'\Gamma = 1 + \sum_n g^{2n}\mu^{n\varepsilon}Z^{(n)}(\mu^2),$$

$$Z^{(n)}(\mu^2) = KR'\Gamma^{(n)}|_{\tau=\mu^2}\,.$$

Because Z is dimensionless we obtain: $Z^{(n)}(\mu^2) = \mu^{-n\varepsilon}Z^{(n)}(1)$
That is why the following relationship is valid for $Z^{(n)}$

$$Z^{(n)} = -\frac{2}{n\varepsilon}\mu^2\partial_{\mu^2}Z^{(n)} = -\frac{2}{n\varepsilon}\mu^2\partial_{\mu^2}KR'\Gamma^{(n)},$$

On the basis of this expression we'll define $\mathcal{N}$ and $\mathcal{J}$ operations

Definition $\mathcal{N}$ и $\mathcal{J}$ operations

$$Z^{(n)} = -\frac{2}{n\varepsilon}\mu^2\partial_{\mu^2}KR'\Gamma^{(n)} = \frac{2}{n\varepsilon}\left(\mathcal{N}\Gamma^{(n)} + \mathcal{J}\Gamma^{(n)}\right)$$

$$\mathcal{N}\Gamma^{(n)} \equiv -KR'\mu^2\partial_{\mu^2}\Gamma^{(n)}$$

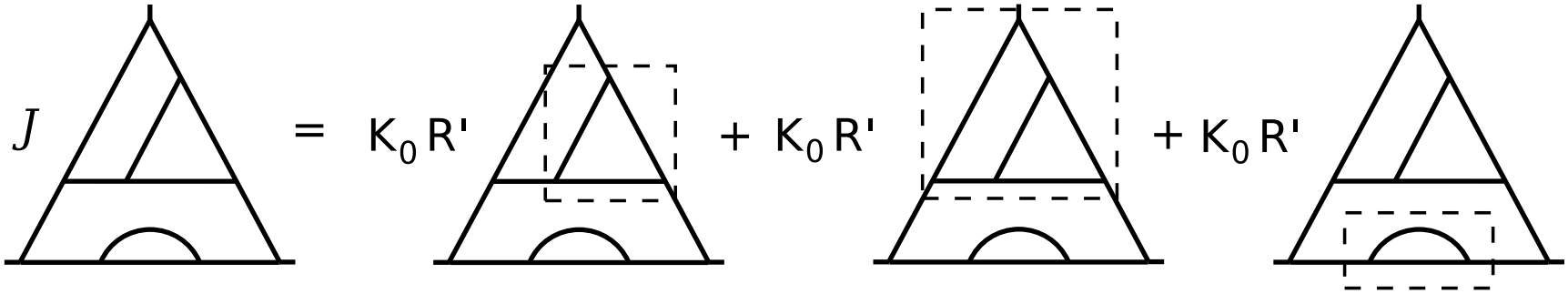
$$\mathcal{J}\Gamma^{(n)} \equiv K\left(R'\mu^2\partial_{\mu^2} - \mu^2\partial_{\mu^2}R'\right)\Gamma^{(n)}.$$

$\mathcal{N}\Gamma^{(n)}$ is finite at $\varepsilon = 0$, because derivative ∂_{μ^2} removes the superficial divergence; R' removes subgraph divergences.

$\mathcal{J}\Gamma^{(n)}$ is an high order pole part.

Simplification of $\mathcal{J}$ operation

Previously introduced representation



can be obtained from

$$\mathcal{J}\Gamma^{(n)} = K\left(R'\mu^2\partial_{\mu^2} - \mu^2\partial_{\mu^2}R'\right)\Gamma^{(n)}$$

by means of rearrangement of terms, and can be proved by method of induction.

Resultant expression for $\mathcal{N}\Gamma^{(n)}$

$G(k) =$ $= \dfrac{1}{k^2 + \mu^2}$,

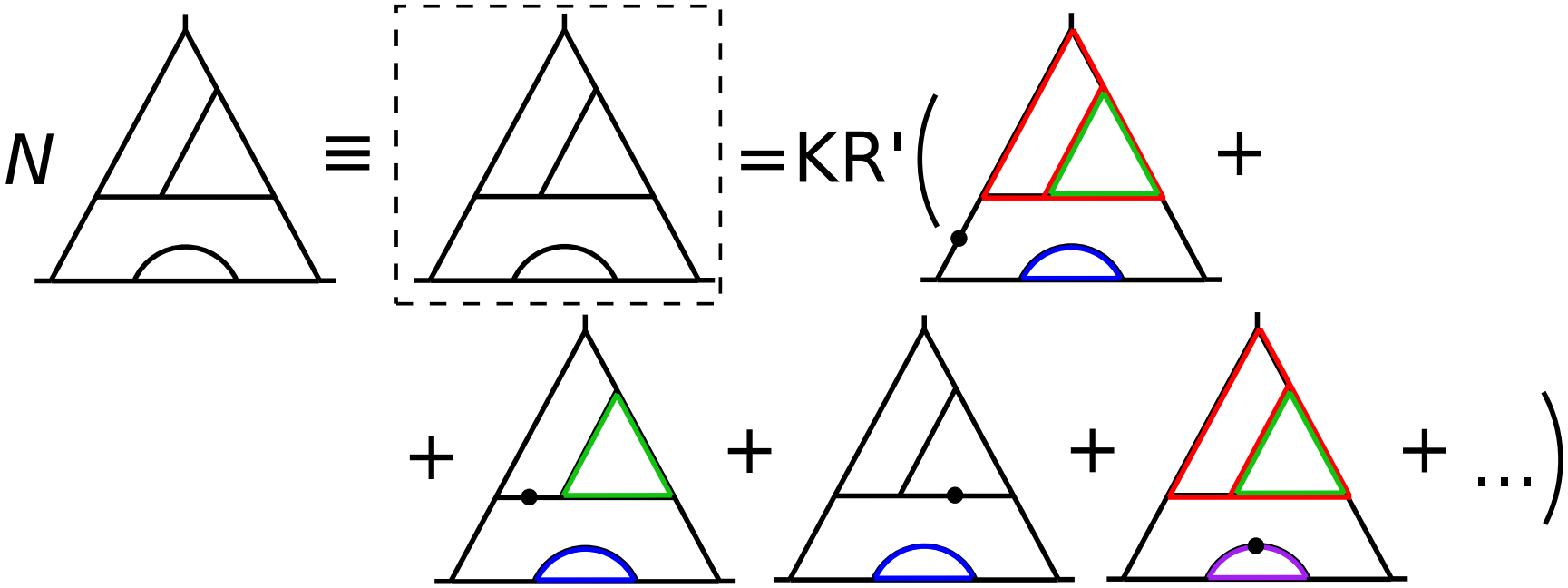
$-\partial_{\mu^2} G(k) = \dfrac{1}{(k^2 + \mu^2)^2} = G^2(k) =$,

Therefore representation

$$\mathcal{N}\Gamma^{(n)} \equiv -KR'\mu^2 \partial_{\mu^2} \Gamma^{(n)}$$

is correspondent to the sum of all kindes of insertions of unity vertex into one of the graph's line
This results in the formerly declared representation for $\mathcal{N}$ operation

Genuine n-loop term $\mathcal{N}\gamma$ in representation $KR'\gamma = \frac{2}{n\epsilon}(\mathcal{N}\gamma + \mathcal{J}\gamma)$,



- Insertion of the unity vertex removes the superficial divergence
- Insertion of the unity vertex changes the structure of divergences
- KR' makes the expression finite at $\epsilon = 0$

The forthcoming part of the report is devoted to the problems connected with representation of $\mathcal{N}\gamma$ in the form convenient for numerical calculations

It is necessary to express the result of R-operation as the unified expression, which does not contain divergences

The conventional procedure of R-operation (graph minus counterterms) results in poor convergent integrals (due to cancellation of large numerical terms)

We'll propose the way of R-operation representation, which allows to express $\mathcal{N}\gamma$ **as an well convergent integral**

The problem of cancellation of large numerical contributions

The standard approach in writing the expression for R-operation **«graph minus counterterms»** guarantees a finiteness of the value $R\gamma$ due to cancellation of pole contributions of separate terms Example

$$\int_0^\infty dk\,\frac{k^3}{(k^2+\tau)}\left[\frac{1}{(k^2+\tau)}-\frac{1}{k^2}\right]=\int_0^\infty dk\,\frac{-k\tau}{(k^2+\tau)^2}.$$

Reduction to common denominator for graphs in high orders of loop expansion leads to enormously large expressions and has no practical usage.

Constructive realization of R-operation

- O. I. Zavialov: Renormalized Feynman Diagrams, Moscow, Nauka, 1979, 317 p.

$$R = (1-K)R' = (1-K)\prod_j (1-K^{(j)}).$$

$(1-K^{(j)})$ - substraction operation affecting j-th divergent subgraph

$(1-K)$ - substraction operation affecting the graph as a whole

In our renormalization scheme $(1-K)$ is substraction of the initial part of the momentum series, which can be presented as

$$(1-K)F(k) = F(k) - \sum_{m=0}^{n} \frac{k^m}{m!} F^{(m)}|_{k=0} = \frac{1}{n!}\int_0^1 da(1-a)^n \partial_a^{n+1} F(ak).$$

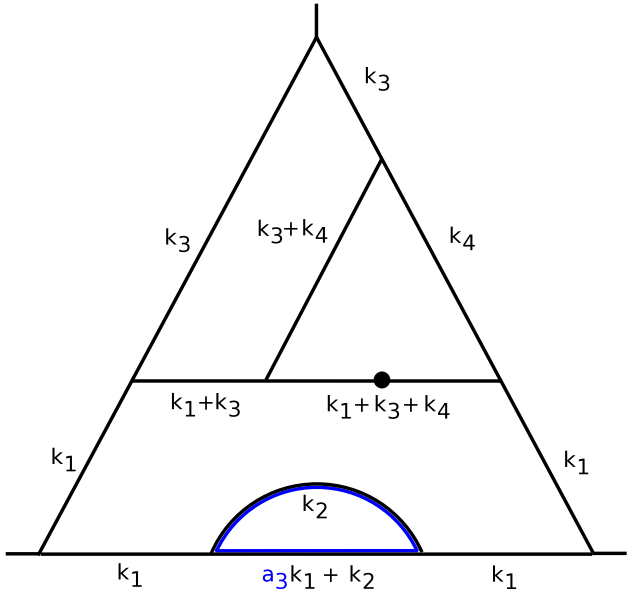
$$R\Gamma = \prod_i \frac{1}{n_i!}\int_0^1 da_i (1-a_i)^{n_i} \partial_{a_i}^{n_i+1} \Gamma(\{a\}),$$

$n_i \geq 0$, – dimension of subgraph , a_i – parameter of momentum rescaling inside i-th subgraph (only momenta which are external to i-th subgraph are rescaled)

i - counts subgraphs and graph as a whole

Example of calculation of genuine terms $\mathcal{N}\Gamma^{(n)}$

$\Gamma_1^{(4)}(\{a\},\{\mathbf{k}\}) =$

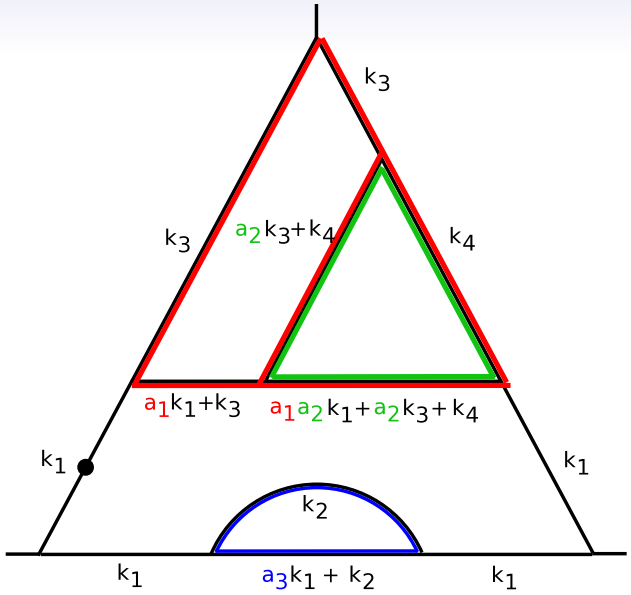


$$\mathcal{N}\Gamma_1^{(4)} = \frac{1}{2}\int_0^1 d\,a_3\,(1-a_3)^2\,\partial_{a_3}^3 \int d\,\{\mathbf{k}\}\;\Gamma_1^{(4)}(\{a\},\{\mathbf{k}\})$$

$$\Gamma_1^{(4)}(\{a\},\{\mathbf{k}\}) = \frac{1}{F_1(\{a\},\{\mathbf{k}\})}$$

$$F_1 = (k_1^2+1)^5(k_2^2+1)(k_3^2+1)^2(k_4^2+1)\left[(\mathbf{k_1}+\mathbf{k_3}+\mathbf{k_4})^2+1\right]\cdot$$
$$\cdot\left[(a_3\mathbf{k_1}+\mathbf{k_2})^2+1\right]\left[(\mathbf{k_1}+\mathbf{k_3})^2+1\right]\left[(\mathbf{k_3}+\mathbf{k_4})^2+1\right]$$

$$\Gamma_2^{(4)}(\{a\},\{\mathbf{k}\}) =$$

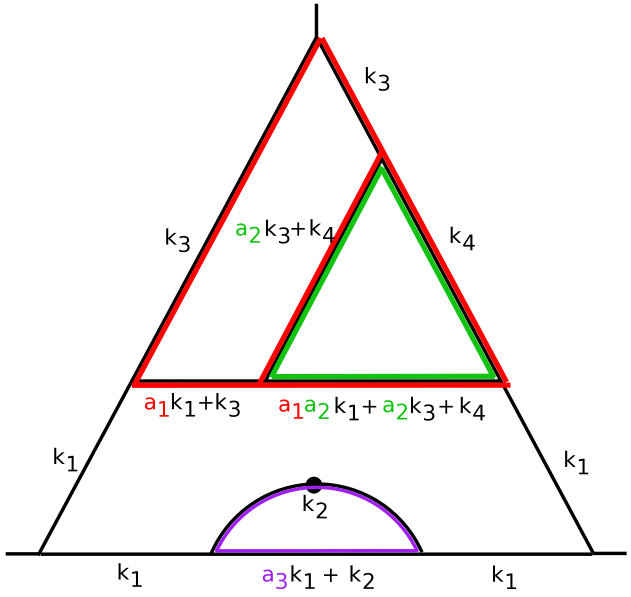


$$\mathcal{N}\Gamma_2^{(4)} = \frac{1}{2}\int_0^1 d\,a_1\,\partial_{a_1}\int_0^1 d\,a_2\,\partial_{a_2}\int_0^1 d\,a_3\,(1-a_3)^2\,\partial_{a_3}^3\int d\,\{\mathbf{k}\}\;\Gamma_2^{(4)}(\{a\},\{\mathbf{k}\})$$

$$\Gamma_2^{(4)}(\{a\},\{\mathbf{k}\}) = \frac{1}{F_2(\{a\},\{\mathbf{k}\})}$$

$$F_2 = (k_1^2+1)^5(k_2^2+1)(k_3^2+1)^2(k_4^2+1)\left[(a_1a_2\mathbf{k_1}+a_2\mathbf{k_3}+\mathbf{k_4})^2+1\right]\cdot$$
$$\cdot\left[(a_3\mathbf{k_1}+\mathbf{k_2})^2+1\right]\left[(a_1\mathbf{k_1}+\mathbf{k_3})^2+1\right]\left[(\mathbf{k_3}+\mathbf{k_4})^2+1\right]$$

$$\Gamma_3^{(4)}(\{a\},\{\mathbf{k}\}) =$$



$$\mathcal{N}\Gamma_3^{(4)} = \int_0^1 d\,a_1\,\partial_{a_1} \int_0^1 d\,a_2\,\partial_{a_2} \int_0^1 d\,a_3\,\partial_{a_3} \int d\,\{\mathbf{k}\}\;\Gamma_3^{(4)}(\{a\},\{\mathbf{k}\})$$

$$\Gamma_3^{(4)}(\{a\},\{\mathbf{k}\}) = \Gamma_2^{(4)}(\{a\},\{\mathbf{k}\})$$

- Proposed realization of R operation produces **only convergent integrals**
- Such an integrals can be easily calculated using Monte Carlo method
- This allows us to calculate genuine part of any particular graph.
- If we know genuine part of any graph then we can construct R' for any graph
- This leads to the fact that we can calculate renormalization constants in any order of ϵ-expansion. Order of ϵ-expansion is limited only by available computational power.

We performed calculation of critical exponent η up to the 4-th order of ϵ- expansion

$$\eta = -0.1111\epsilon - 0.0588\epsilon^2 + 0.0439\epsilon^3 - 0.081\epsilon^4 + O(\epsilon^5)$$

precision of integral calculation was 10^{-4}
Analytical calculations of the 3rd order of ϵ-expansion:

- O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane, J.Phys A: Math Gen. 13 (1980) L245-L251
- O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane, J.Phys A: Math Gen. 14 (1981) 2391-2413

$$\eta = -\frac{\epsilon}{9} - \frac{43\epsilon^2}{3^6} + \epsilon^3\left(-\frac{8375}{2^2 3^{10}} + \frac{16\zeta(3)}{3^5}\right) + O(\epsilon^4) \approx$$

$$\approx -0.11111\epsilon - 0.05898\epsilon^2 + 0.04367\epsilon^3 + O(\epsilon^4)$$

Thank you for attention!

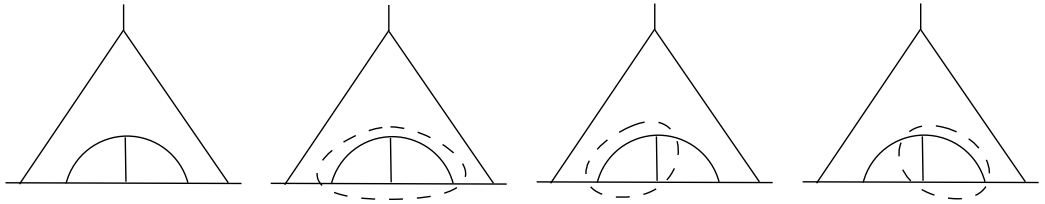
Константы Z_i в используемой схеме ренормировки

$$Z_2 = 1 + K_2 R' \Gamma_2 \,, \qquad Z_3 = 1 + K_0 R' \bar{\Gamma}_3 \,, \qquad \bar{\Gamma}_3 \equiv \Gamma_3 / (g \mu^{\varepsilon/2}) \,,$$

$$Z_1 = 1 + \frac{1}{\mu^2} \left(R' \Gamma_2 |_{\otimes} - R' \Gamma_2 |_0 \right) .$$

Изложенный метод непосредственно применим для Z_3.
С учетом коммутативности операций R' и $\partial_p^2 ... |_{p=0}$ его можно использовать и для Z_2.
Константу Z_1 можно заменить на эквивалентную константу ренормировки Z_ϕ составного оператора $\phi = \varphi^2$, для которой рассматриваемая техника также применима.

Замечание о перекрывающихся подграфах.
Представление справедливо для R операции, но не для R' операции (нельзя “сократить“ на $(1-K)$). Дело в том, что если в диаграмме имеются перекрывающиеся существенные подграфы (в нашем случае вершинные), например, $\Gamma^{(i)}$ и $\Gamma^{(j)}$, то в определение R' операции не входит произведение операций $K^{(j)}K^{(i)}$, присутствующее в . Объединение существенных подграфов дает существенный подграф $\Gamma^{(l)} = \Gamma^{(i)} \bigcup \Gamma^{(j)}$ (двухвостый) или диаграмму в целом. В первом случае действующая на этот подграф операция $(1-K^{(l)})$ обращает “лишнюю” величину $K^{(i)}K^{(j)}\Gamma^{(l)}$ в ноль, так как последняя не зависит от импульса, если же объединение дает диаграмму в целом, то “лишнее” произведение исчезает только для R операции. Пример:



Резюме:

Контрчлены полностью определяются существенной частью диаграмм, вычисление которой включает следующие шаги:

- Построение диаграммы и определение комбинаторного множителя;
- Расстановка точки на линиях и нахождение существенных подграфов для каждой из расстановок;
- Растяжение втекающих в подграфы импульсов и запись подынтегрального выражения.
- Выполнение численного интегрирования по импульсам и параметрам растяжения.

для вычисления была написана программа осуществляющая

- построение диаграмм
- Вычисление для них $\mathcal{N}\gamma$
- Построение по $\mathcal{N}\gamma$ констант ренормировок Z
- Вычисление РГ-функций