# Effective action for the Regge processes in QCD and in gravity

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# 1 Regge ansatz for amplitudes

Regge kinematics for elastic scattering

$$s = 4E^2 >> -t = \vec{q}^2$$

$t$-channel partial wave expansion

$$A^p(s,t) = s \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left((-s)^\omega - ps^\omega\right) f^p_\omega(t)\,, \;\; p = \pm 1$$

Regge pole hypothesis

$$f^p_\omega(t) = \frac{\gamma^2(t)}{\omega - \omega(t)}$$

Linear Regge trajectories

$$\omega_p(t) = \Delta - \alpha' \vec{q}^2$$

# 2 Mandelstam cuts

Regge asymptotics

$$A^p_{Regge}(s,t) = \xi_p(t)\, s^{1+\omega_p(t)}\, \gamma^2(t)\,,\ \ \xi_p(t) = e^{-i\pi\omega_p(t)} - p$$

Mandelstam cut contribution

$$A^p_{Mand}(s,t) = \xi_p\, s \int \frac{d^2k}{(2\pi)^2} \Phi^2(k, q-k) s^{\omega_{p_1}(-k^2)} s^{\omega_{p_2}(-(q-k)^2)}$$

Gribov signature conservation rule

$$p = p_1\, p_2$$

Total cross sections and Pomeron

$$\sigma = \frac{1}{s}\, \Im A(s.0)\,,\ \ A(s,t) \approx is\, s^{\Delta_P - \alpha'_P q^2}\,,\ \ \Delta_P << 1$$

# 3 Gribov Pomeron calculus

Multi-particle unitarity conditions in $t$-channel

$$\Im_t f_\omega(t) \sim \sum_n \int d\Omega_n |f_\omega^{(n)}|^2$$

Separation of particle clusters in their rapidities

$$0 < y_1 < y_2 < ... < y_k < \ln s\,,\ 1 << y_k - y_{k-1} << \ln s$$

Non-relativistic reggeon propagators

$$G_0 = \frac{1}{E + \Delta - \frac{k^2}{2m}}\,,\ E = -\omega\,,\ \alpha' = \frac{1}{2m}$$

Gribov effective Pomeron action

$$S = \int dy\, d^2\rho \left( \phi^*(\partial_y - \Delta)\phi + \frac{1}{2m}\,|\partial_\mu \phi|^2) + i\lambda(\phi^*\phi^2 + \phi\phi^{*2}) + ... \right)$$

# 4 Gluon reggeization in QCD

Leading logarithmic approximation

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\, s^{\omega(t)}\,,\ \alpha_s \ln s \sim 1\,,\ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

Multi-Regge amplitudes in LLA (F.,K.,L. (1975))

$$M_{2\to 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT_{c_2c_1}^{d_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT_{c_{n+1}c_n}^{d_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi} \left( \ln \frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon} \right)\,,\ C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

# 5 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\ ,\ \sigma_t \sim s^{\Delta}\ ,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

BFKL Hamiltonian

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$\rho_{12} = \rho_1 - \rho_2\ ,\ \rho_r = x_r + iy_r\ ,\ \Delta = 4\alpha N_c\,\ln 2\,/\pi$$

Möbius invariance and eigenvalues (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\ m = \gamma + n/2\,,\ \widetilde{m} = \gamma - n/2\,,\ \gamma = 1/2 + i\nu\,,$$

$$E = \psi(m) + \psi(1-m) + \psi(\widetilde{m}) + \psi(1-\widetilde{m}) - 4\psi(1)$$

# 6 Effective action in QCD

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|} \,, \; |y - y_0| < \eta \,, \; \eta << \ln \, s$$

Gluon and Reggeized gluon fields

$$v_\mu(x) = -iT^a v^a_\mu(x) \,, \; \partial_\mp A_\pm = 0 \,, \, \delta A_\pm(x) = 0$$

Effective action for reggeized gluons (L., 1995)

$$S = \int d^4x \left( L_{QCD} + Tr(V_+ \partial^2_\mu A_- + V_- \partial^2_\mu A_+) \right) \,,$$

$$V_+ = -\frac{1}{g} \partial_+ \, P \exp \left( -\frac{g}{2} \int_{-\infty}^{x^+} v_+(x') d(x')^+ \right) = v_+ - g v_+ \frac{1}{\partial_+} v_+ + ...$$

# 7 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. and C.,C. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1) \;-\; \frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

# 8 Pomeron and reggeized graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the energy-momentum conservation

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling asymptotics for $\Delta$ (KLOV, BPST)

$$\gamma = -\sqrt{2\pi(j-2)}\,\hat{a}^{1/4}\,,\ j = 2 - \Delta\,,\ \Delta = \frac{1}{2\pi}\,\hat{a}^{-1/2}$$

# 9 Reggeized gluons and gravitons

RGG vertices for QCD and gravity (L. (1982))

$$\Gamma_A^{\mu\mu'} = \delta^{\mu\mu'} - \frac{P_A^{\mu'} p_B^{\mu} + P_{A'}^{\mu} p_B^{\mu'}}{p_A p_B} - q^2 \frac{p_B^{\mu} p_B^{\mu'}}{2(p_A p_B)^2}\,, \quad \Gamma_A^{\nu\nu'\mu\mu'} = \Gamma_A^{\mu\mu'} \Gamma_A^{\nu\nu'}$$

RRG vertex for QCD (L. (1975))

$$\Gamma_\mu = \left( -q_1^\perp - q_2^\perp + p_A \left( \frac{k p_B}{p_A p_B} + \frac{q_1^2}{k p_A} \right) - p_B \left( \frac{k p_A}{p_A p_B} + \frac{q_2^2}{k p_B} \right) \right)_\mu$$

RRG vertex for gravity (L. (1982))

$$\Gamma_{\mu\nu} = \frac{1}{2} \Gamma_\mu \Gamma_\nu - \frac{1}{2} q_1^2 q_2^2 \left( \frac{p_A}{p_A k} - \frac{p_B}{p_B k} \right)_\mu \left( \frac{p_A}{p_A k} - \frac{p_B}{p_B k} \right)_\nu$$

Graviton Regge trajectory (L. (1982))

$$\omega = -q^2 \frac{\kappa^2}{8\pi^3} \int \frac{d^2 k}{k^2 (q-k)^2} \left( (k_\perp, q - k_\perp)^2 \left( \frac{1}{k_\perp^2} + \frac{1}{(q - k_\perp)^2} \right) - q^2 \right)$$

# 10 Effective action for gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|} \, , \; |y - y_0| < \eta \, , \; \eta << \ln \, s$$

Metric tensor and reggeized graviton fields

$$d^2 S = \sum_{\mu\nu} g_{\mu\nu} \, dx^\mu \, dx^\nu \, , \, \delta A^{\pm\pm}(x) = \partial_\pm A^{\pm\pm} = 0$$

Effective action for reggeized gravitons

$$S = \int d^4 x \left( -\frac{\sqrt{-g}}{2\kappa} \, R + j_{++} \, \partial_\mu^2 A^{++} + j_{--} \, \partial_\mu^2 A^{--} \right)$$

Effective current (from L. (1982))

$$j_{++} = g_{++} + (\partial_\sigma g_{++}) \frac{1}{4\partial_+^2} \partial_\sigma g_{++} - (\partial_\sigma g_{++}) \frac{1}{\partial_+} g_{+\sigma} - g_{+\sigma}^2 + ...$$

# 11 General covariance

Metric tensor coordinate transformation

$$\delta g_{\mu\nu} = D_\mu \chi_\nu + D_\nu \chi_\mu \,, \;\; \delta j_{\pm\pm} = 0$$

Gravitation field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \;\; g^{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + h_{\mu\rho} h_{\rho\nu} - ...$$

New variables

$$X_{\rho+} = h_{\rho+} - \frac{1}{2} \frac{\partial_\rho}{\partial_+} h_{++} \,, \;\; y_{\rho\sigma} = h_{\rho\sigma} - 2 \frac{\partial_\rho}{\partial_+} X_{\sigma+}$$

Perturbative expansion from the covariance condition

$$j_{++} = h_{++} - X_{\sigma+}^2 + X_{\rho+} X_{\sigma+} h_{\rho\sigma} - X_{\sigma+} \frac{\partial_\sigma}{\partial_+} X_{\rho+}^2 + ...$$

# 12 "Eikonal" contribution

Scattering off the gravitation field $h_{++}$

$$A_2 = \frac{1}{(p+k_1)^2} + \frac{1}{(p+k_2)^2} \approx -\frac{(k_1 k_2)}{2(pk_1)(pk_2)}\,, \;\; A_3 =$$

$$\frac{k_3(k_1+k_2)}{4pk_3\,p(k_1+k_2)}\,\frac{k_1k_2}{pk_1\,pk_2} + \frac{k_2(k_1+k_3)}{4pk_2\,p(k_1+k_3)}\,\frac{k_1k_3}{pk_1\,pk_3} + \frac{k_1(k_2+k_3)}{4pk_1\,p(k_2+k_3)}\,\frac{k_2k_3}{pk_2\,pk_3},$$

$$A_4 = -\frac{k_1k_2}{16\,pk_1\,pk_2}\,\frac{k_4(k_1+k_2+k_3)}{pk_4\,p(k_1+k_2+k_3)}\,\frac{k_3(k_1+k_2)}{pk_3\,p(k_1+k_2)}$$

$$-\frac{(k_3+k_4)(k_1+k_2)}{32\,p(k_3+k_4)\,p(k_1+k_2)}\,\frac{k_1k_2}{pk_1\,pk_2}\,\frac{k_3k_4}{pk_3\,pk_4} + permutations$$

"Eikonal" contribution expansion

$$j^{eik}_{++} = h_{++} - X^2_{\sigma+} - X_{\sigma+}\frac{\partial_\sigma}{\partial_+}X^2_{\rho+} - X_{\mu-}\frac{\partial_\mu}{\partial_-}X_{\sigma-}\frac{\partial_\sigma}{\partial_-}X^2_{\rho-} - \frac{1}{4}\left(\frac{\partial_\sigma}{\partial_-}X^2_{\rho-}\right)^2 + ...$$

# 13 Equations of motion for currents

Effective currents (reggeon wave functions)

$$J_{\pm} = h_{\pm\pm} - \frac{1}{\partial_{\pm}}\, j_{\pm\pm}$$

Fan equation for the "eikonal" contribution

$$\left(\partial_{+} - X_{\sigma+}\partial_{\sigma}\right)\, J_{+}^{eik} = X_{\mu+}^{2} + \frac{1}{4}\left(\partial_{\rho} J_{+}^{eik}\right)^{2}$$

New effective currents

$$j_{\pm\pm} = \partial_{\pm} j^{\mp}\,,\ \Delta S = \frac{1}{2\kappa}\int d^{2}k_{\perp}\left(\int dx^{-}\, j^{-}_{x^{+}\to\infty}\partial_{\mu}^{2}A^{++} + \int dx^{+}\, j^{+}_{x^{-}\to\infty}\partial_{\mu}^{2}A^{--}\right)$$

Light front equation in the covariant form

$$\left(\frac{1}{2}\, D^{\rho} j^{+} + g^{\rho+}\right)\left(\frac{1}{2}\, D_{\rho} j^{+} + g^{+}_{\rho}\right) = 0$$

# 14 Discussion

1. Gribov Pomeron calculus.
2. Locality in the rapidity space.
3. Effective action for reggeized gluons.
4. Pomeron-reggeized graviton correspondence in $N = 4$ SUSY.
5. Effective vertices for regeized graviton interactions.
6. Effective action for reggeized gravitons.
7. "Eikonal" contributions to action.
8. "Fan" equation for effective currents.