Study of the anomalous kinetics of the annihilation reaction $A + A \to \varnothing$

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- broad class of chemical reactions $A + A \xrightarrow{\lambda_0} \varnothing$

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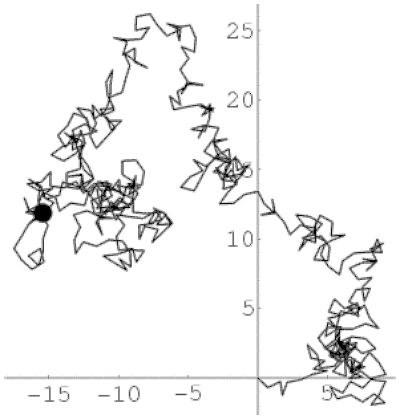

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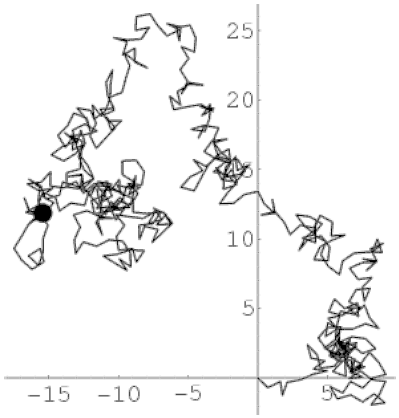

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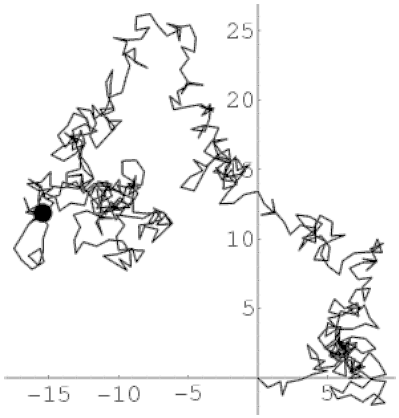

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- What's the value of decaying exponent α, $n(t) \xrightarrow{t\rightarrow\infty} t^{-\alpha}$?

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- let $d \leq 2$, in this case the diffusion is recurrent(Pólya theorem 1921)
  R.M.S. of displacement $r(t) \sim (Dt)^{1/2}$ and "sweep" volume $V(t) \sim r(t)^d$,
  what leads to $n(t) \sim (Dt)^{-d/2} = (Dt)^{-(1+\Delta)}$
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- for $d > 2$ $V(t) \sim t \Rightarrow n(t) \sim t^{-1}$

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  (b) $\mathbf{v}(\mathbf{x},t)$ generated by stochastic Navier Stokes equation

$$\partial_t \mathbf{v} + (\mathbf{v}.\nabla)\mathbf{v} = \nu_0\nabla^2\mathbf{v} - \nabla p + \mathbf{f}^v \tag{2}$$

$\mathbf{f}^v$ - random force

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- starting point - master equation

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- $\Rightarrow$ suggestion of using second quantization method

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- 'boson'-like operators (no i and $\hbar$)

$$[\psi(\mathbf{x}), \psi^{\dagger}(\mathbf{x}^{'})] = \delta(\mathbf{x} - \mathbf{x}^{'}) \tag{5}$$

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- discrete model (particles on the lattice) $\overset{\text{Doi formalism}}{\Longrightarrow}$ continuous model
  L. Peliti J. Physique **46**, 1469 (1985)

Field model for chemical reaction

- in this formulation mean values could be obtained via

$$\langle A(t)\rangle = \langle 0|\mathrm{e}^{\int d\mathbf{x}} A(\psi^{+}\psi)\mathrm{e}^{-\hat{H}t}|\phi(0)\rangle \tag{10}$$

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- action S_1 could be written as

$$S_1 = -\int_0^\infty dt \int d\mathbf{x}\, \{\psi^+\partial_t\psi + \psi^+\nabla(\mathbf{v}\psi) - D_0\psi^+\nabla^2\psi + \lambda_0 D_0[2\psi^+ + (\psi^+)^2]\psi^2\} + n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x},0) \tag{12}$$

Toy model

- describe advection of passive scalar $\psi(t, \mathbf{x})$

$$\partial_t \psi + (\mathbf{v}.\nabla)\psi = D_0 \partial^2 \psi + f$$

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- in real problem $\mathbf{v}(x)$ satisfies NS eq. but now its properties are given as follows

$$\langle \mathbf{v}(x) \rangle = 0, \qquad \nabla.\mathbf{v} = 0, \qquad m \sim 1/L$$

$$\langle v_i(x) v_j(x') \rangle = \frac{g_0 D_0 \delta(t - t')}{(2\pi)^d} \int d\mathbf{k} P_{ij}(\mathbf{k}) \theta(k - m) k^{-d-2\epsilon} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}$$

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- the total action

$$S = S_1 + S_{\mathbf{v}}$$
$$\text{where} \qquad S_{\mathbf{v}} = -\int\int d\mathbf{x}dt\, d\mathbf{x}'dt' \frac{\mathbf{v}(\mathbf{x},t) D_v^{-1} \mathbf{v}(\mathbf{x}',t')}{2}$$

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- model is multiplicatively renormalizable (divergences in $\langle \psi^+ \psi \rangle_{\text{1-ir}}$, $\langle \psi^+ \psi^2 \rangle_{\text{1-ir}}$ and $\langle (\psi^+)^2 \psi^2 \rangle_{\text{1-ir}}$)

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- renormalization relations in minimal subtraction scheme

$$D_0 = D Z_D, \quad g_0 = g \mu^{2\epsilon} Z_g, \quad Z_g = Z_D^{-1}, \quad \lambda_0 = \lambda \mu^{-2\Delta} Z_D^{-1} Z_\lambda$$

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- two-loop calculation employing double expansion (ϵ, Δ) with using MS scheme gives

$$Z_2^{(1)} = -\frac{\overline{g}(d-1)}{4d\epsilon}, \qquad \overline{g} = \frac{gS_d}{(2\pi)^d}$$

$$Z_4^{(1)} = -\frac{\overline{\lambda}}{2\Delta}$$

$$Z_4^{(2)} = \frac{d-1}{4d}\frac{\overline{g\lambda}}{2\epsilon - 2\Delta}\left(-\frac{1}{\Delta} + \frac{2}{d+2}\,{}_2F_1(1,1,2+\frac{d}{2};1)\right) + \frac{\overline{\lambda}^2}{4\Delta^2}$$

where $Z_\alpha^{(n)}$ means contributions $\sim g^n$ to Z_α

Fixed points of toy model

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- from Z_α beta functions $\beta_g = \mu \frac{\partial g}{\partial \mu}|_0$ and anomalous dimensions $\gamma_a = \mu \frac{\partial \ln Z_a}{\partial \mu}|_0$ could be exctracted
- results to the two-loop order

$$\beta_g = g(-2\epsilon + \gamma_D), \qquad \beta_\lambda = \lambda(2\Delta - \gamma_\lambda + \gamma_D)$$
$$\gamma_D = \frac{d-1}{2d}\overline{g}, \qquad \gamma_\lambda = -\overline{\lambda} \qquad (\overline{g} = gS_d/(2\pi)^d) \tag{13}$$

- four possible fixed points with region of stability (ϵ - deviation from Kolmogorov scaling and $2\Delta = d - 2$)
  - (I) $g^* = \lambda^* = 0$ with $\epsilon < 0, \Delta > 0$
  - (II) $g^* = 0, \lambda^* = -4\pi\Delta$ with $\epsilon < 0, \Delta < 0$
  - (III) $g^* = \frac{4d\epsilon}{\overline{S_d}(d-1)}, \lambda^* = 0$ with $\epsilon > 0, \Delta > -\epsilon$
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- not very good model - same results as in one loop-order

stochastic Navier-Stokes equation

- eq. for fluctuating part of velocity field

$$\partial_t \mathbf{v} + (\mathbf{v}.\nabla)\mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - \nabla p + \mathbf{f}^v \tag{14}$$

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$$\begin{aligned} S_{NS} &= \frac{1}{2}\int dt\, d\mathbf{x}\, d\mathbf{x}'\, \tilde{\mathbf{v}}(\mathbf{x},t).\tilde{\mathbf{v}}(\mathbf{x}',t) d_f(|\mathbf{x}-\mathbf{x}'|) + \\ &\int dt\, d\mathbf{x}\, \tilde{\mathbf{v}}.[-\partial_t \mathbf{v} - (\mathbf{v}.\nabla)\mathbf{v} + \nu_0 \nabla^2 \mathbf{v}] \end{aligned} \tag{15}$$

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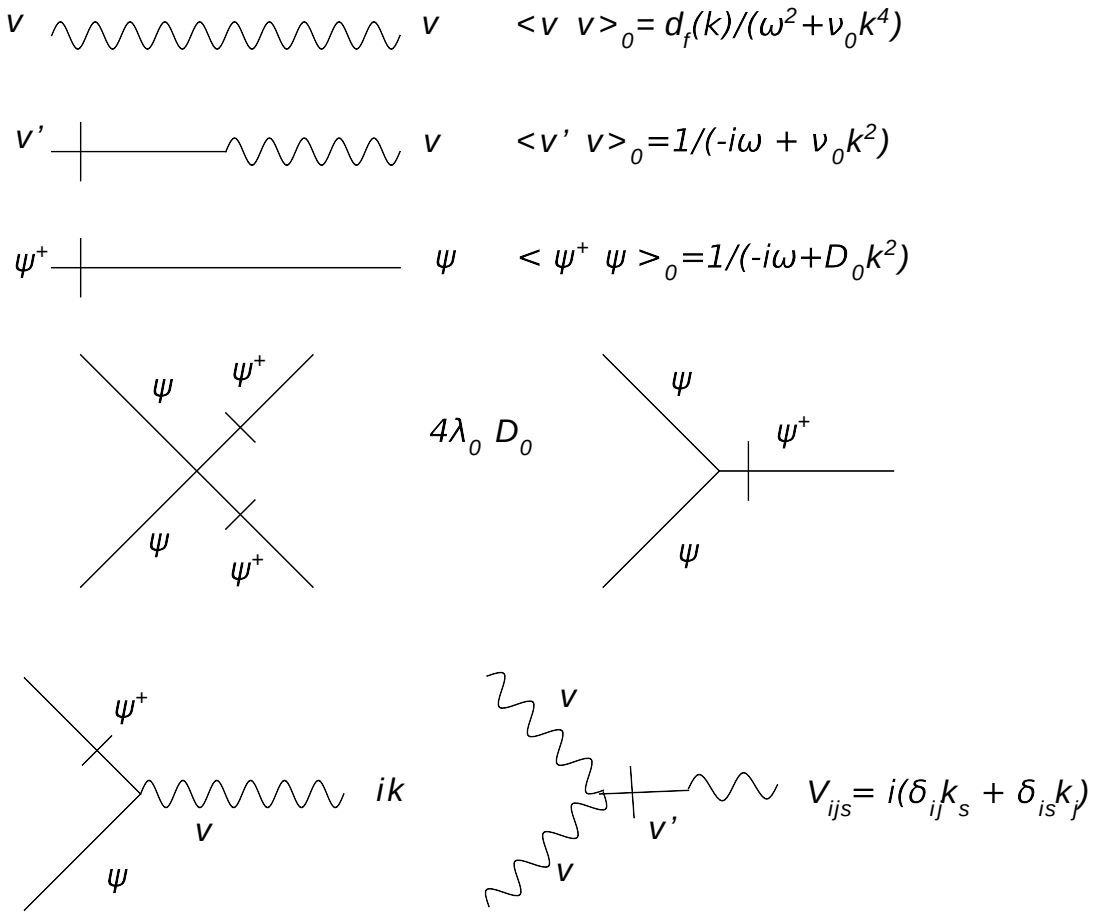
- correlator in Fourier representation $d_f(k) = g_{10}\nu^3 k^{4-d-2\epsilon} + g_{20}\nu^3 k^2$

Definition of propagators and interaction vertices

- diagrammatic representation



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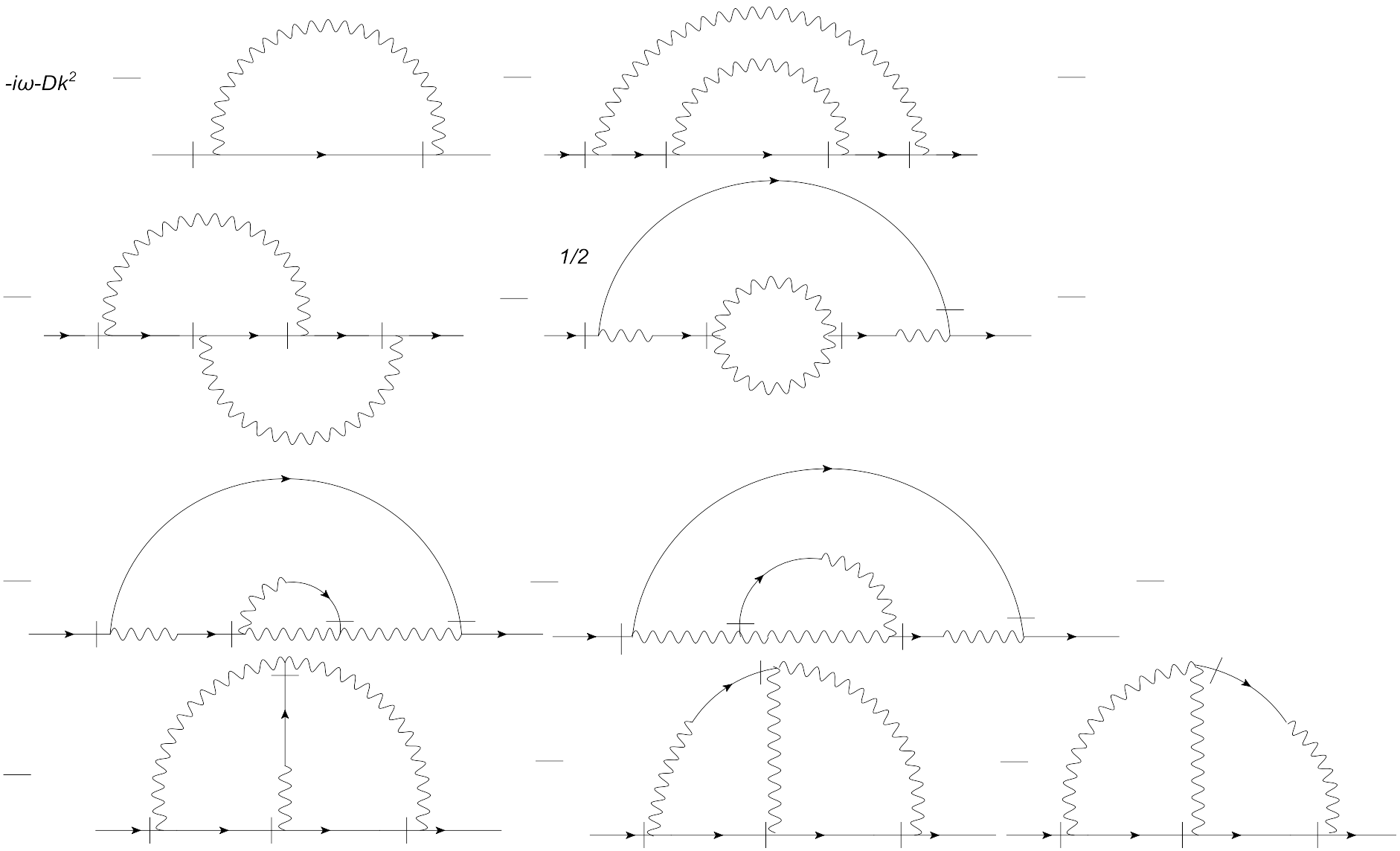
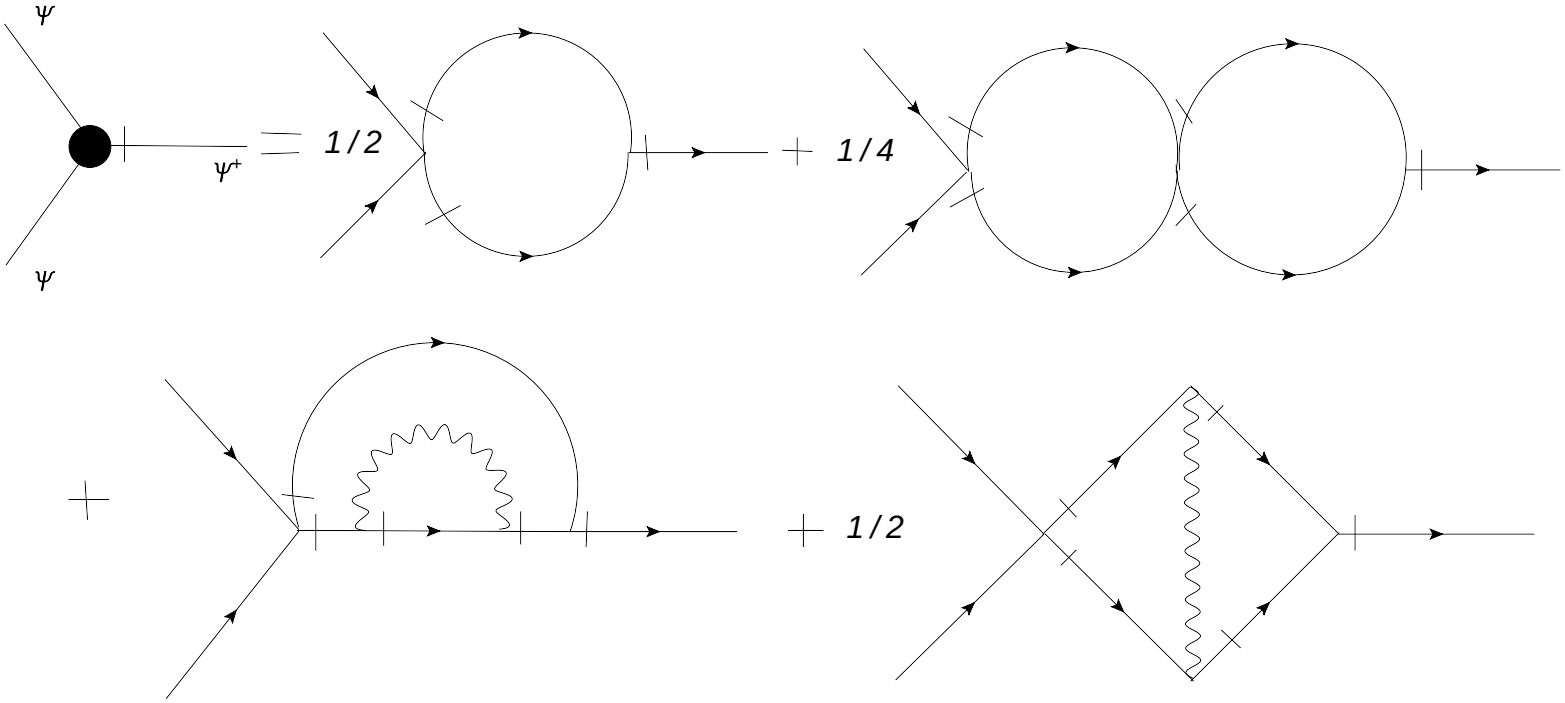
- the form of beta functions could be directly obtained from definition

$$\beta_{g_1} = g_1(-2\epsilon + 3\gamma_1), \quad \beta_{g2} = (2\Delta + 3\gamma_1 - \gamma_3)$$
$$\beta_\lambda = \lambda(2\Delta - \gamma_4 + \gamma_2), \quad \beta_u = u(\gamma_1 - \gamma_2)$$

Dyson equation for propagator $\Delta_{\psi^+\psi} = -\Gamma_{\psi^+\psi}$



Perturbative expansion for $\Gamma_{\psi^+\psi^2}$



Fixed points

$$Z_4 = 1 + \frac{A_3^4 \overline{g_3}}{\epsilon} + \frac{C_{1,3}^4 \overline{g_1 g_3}}{\epsilon^2} + \frac{B_{1,3}^4 \overline{g_1 g_3}}{\epsilon} + + \frac{C_{2,3}^4 \overline{g_2 g_3}}{\epsilon^2} + \frac{B_{2,3}^4 \overline{g_2 g_3}}{\epsilon} + \frac{C_{3,3}^4 \overline{g_3}^2}{\epsilon^2} \tag{16}$$

$$\Delta = \xi\epsilon, \qquad \overline{g} = \frac{g S_d}{(2\pi)^d}$$

$$A_3^4 = -\frac{1}{2\xi}, \quad C_{1,3}^4 = -\frac{1}{16u(1+u)}\frac{1}{\xi(1-\xi)}, \quad C_{2,3}^4 = \frac{1}{32u(1+u)}\frac{1}{\xi^2}$$

$$B_{1,3}^4 = -\frac{1}{16u(1+u)(1-\xi)}\left(-\frac{4(1+u)}{\pi u}\int_{-1}^{1} dz(1-z^2)^{\frac{d-1}{2}} F(z) + \ln\frac{1+u}{2u} + 1 + \frac{2+u}{u}\ln\frac{2+u}{2+2u}\right) = \frac{2\xi}{\xi-1} B_{2,3}^4$$

$$C_{3,3}^4 = \frac{1}{4\xi^2}$$

Results to the two-loop order

decaying exponent of the particle concentration $n(t) \propto t^{-\alpha}$

$$\overline{g}^*_\alpha = \overline{g}^*_{\alpha 1}\epsilon + \overline{g}^*_{\alpha 2}\epsilon^2$$
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Fixed point	α	region of stability

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$Q = 3.788, R = -0.168$

Results and plan for future work

- constructed field-theoretic model of annihilation reaction
- obtained renormalization constans and RG functions to the two-loop order
- studied toy model and compared for the description of advecting field
- solve Callan-Symanzik eq. for $n(t) = \langle \psi(t) \rangle$ to obtain decaying exponents at one - loop order

Thank you for your attention