

Renormalization of nonquasiparton operators in QCD

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BASED ON

V.M. Braun, A.N. Manashov, J. Rohrwild, Nucl. Phys. B807:89-137,2009.

V.M. Braun, A.N. Manashov, J. Rohrwild, Nucl. Phys. B826:235-293,2010.

MQFT, Petergof, October 2010

Twist

Operator Product Expansion

$$J(x)J(0) \sim \sum_N C_N(x^2, \mu^2) \mathcal{O}_N(\mu^2)$$

Twist: $t = \text{dimension} - \text{spin}$:

$$\mathcal{O}_{\mu_1 \dots \mu_N}^{t=2} = \text{Sym } \bar{q} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} q - \text{Traces}$$

$$\mathcal{O}_{\mu_1 \dots \mu_N}^{t=4} = \text{Sym } \bar{q} \gamma_{\mu_1} D_{\mu_2} \dots D^2 \dots D_{\mu_N} q - \text{Traces}$$

Reduced matrix elements have different dimension:

$$\langle P | \mathcal{O}_{\mu_1 \dots \mu_N}^{t=2} | P \rangle = P_{\mu_1} \dots P_{\mu_n} \langle \langle \mathcal{O}_N^{t=2} \rangle \rangle, \quad \langle \langle \mathcal{O}_N^{t=2} \rangle \rangle = [\text{mass}]^0$$

$$\langle P | \mathcal{O}_{\mu_1 \dots \mu_N}^{t=4} | P \rangle = P_{\mu_1} \dots P_{\mu_n} \langle \langle \mathcal{O}_N^{t=4} \rangle \rangle, \quad \langle \langle \mathcal{O}_N^{t=4} \rangle \rangle = [\text{mass}]^2$$

which implies the hierarchy (for “hard” scattering at high energies):

$$\text{Physical observable} \sim \sum_N c_N^{t=2} \langle \langle \mathcal{O}_N^{t=2} \rangle \rangle + \sum_{N,k} c_{N,k}^{t=4} \frac{\langle \langle \mathcal{O}_{N,k}^{t=4} \rangle \rangle}{Q^2} + \dots$$

⇐ **Higher twist effects**

Applications

- deep-inelastic scattering
- exclusive and semi-inclusive reactions, spin physics
 - diffractive electroproduction of vector mesons
 - single spin asymmetries
- flavor physics: B -decays
 - higher twist hadronic wave functions
- form factors, electroproduction of nucleon resonances (CLAS12)

Quasiparmonic and Non-quasiparmonic operators

Higher-twist operators = quasiparmonic + nonquasiparmonic

- **Quasiparmonic operators:**

Bukhvostov, Frolov, Lipatov, Kuraev, 1985 (BFLK)

- multiparticle operators built of “plus” field components

- ⇐ set closed under renormalization

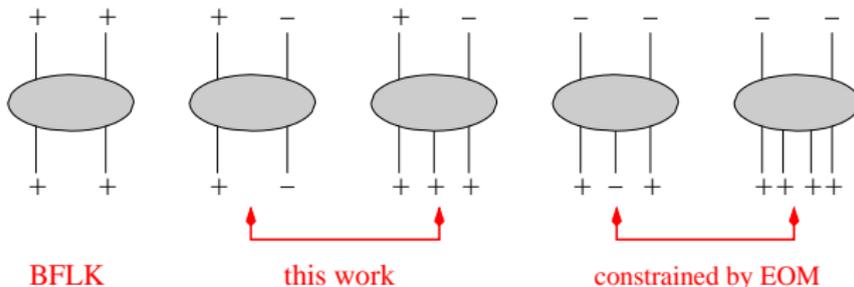
- ⇐ Two-particle structure of renormalization in one loop

- **Nonquasiparmonic operators:**

- all others

- ⇐ mix with quasiparmonic operators

- ⇐ appear starting twist four, e.g. $\bar{\psi}_+ F_{+-} \psi_+$



motivation for this study: to understand renormalization of twist-4 nucleon operators and incorporate some developments in $N = 4$ SUSY:

Beisert, 2004; Beisert, Ferretti, Heise, Zarembo, 2005

- **Methods:**

- Conformal operator basis for arbitrary twist [manifest $SL(2)$ invariance]
- “plus-minus” $2 \rightarrow 2$ kernels by embedding $SL(2, \mathbb{R})$ in $SO(4, 2)$
- $2 \rightarrow 3$ kernels by Lorentz transformation of the BFLK kernels

- **For QCD practitioner:**

- Complete results for operator renormalization up to twist four
- Can be extended to arbitrary twist and maybe beyond LO

Generating function

$$\begin{aligned}
 \mathcal{O}(0, z) &= \bar{q}(0)[0, zn] \not{n} q(zn), & n^2 &= 0 \\
 &= \sum_{N=0}^{\infty} \frac{z^N}{N!} n_{\mu_1} \dots n_{\mu_N} \left[\bar{q} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} q \right] \\
 &= \sum_{N=0}^{\infty} \frac{z^N}{N!} n_{\mu_1} \dots n_{\mu_N} \left[\bar{q} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} q - \text{Traces} \right] \\
 &= \sum_{N=0}^{\infty} \frac{z^N}{N!} n_{\mu_1} \dots n_{\mu_N} \mathcal{O}_{\mu_1 \dots \mu_N}^{t=2}
 \end{aligned}$$

⇐ **Light-ray operator**

$$[0, zn] = \exp \left\{ -ig_s z \int_0^1 du n_{\mu} A^{\mu}(uzn) \right\}$$

Light-ray operators

Example: leading twist

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n] \not{n} q(z_2 n), \quad n^2 = 0$$

RG-equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi} \mathbb{H} \right) [\mathcal{O}(z_1, z_2)]_R = 0$$

where \mathbb{H} is the integral operator

Balitsky, Braun, 1989

$$\begin{aligned} [\mathbb{H} \cdot \mathcal{O}](z_1, z_2) = & 2C_F \left\{ \int_0^1 \frac{d\alpha}{\alpha} \left[2\mathcal{O}(z_1, z_2) - \bar{\alpha} \mathcal{O}(z_{12}^\alpha, z_2) - \bar{\alpha} \mathcal{O}(z_1, z_{21}^\alpha) \right] \right. \\ & \left. - \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \mathcal{O}(z_{12}^\alpha, z_{21}^\beta) - \frac{3}{2} \mathcal{O}(z_1, z_2) \right\} \end{aligned}$$

where $z_{12}^\alpha = z_1(1 - \alpha) + z_2\alpha$, $\bar{\alpha} = 1 - \alpha$.

- \mathbb{H} is invariant under $SL(2, \mathbb{R})$ transformations of the light-ray, $z \rightarrow \frac{az + b}{cz + d}$.
 \Rightarrow **DGLAP, ERBL, GPD**

$$\varphi_{AB}(z_1, z_2) = \langle A | \mathcal{O}(z_1, z_2) | B \rangle$$

$SL(2)$ Algebra

is generated by \mathbf{P}_+ , \mathbf{M}_{-+} , \mathbf{D} and \mathbf{K}_-

$$\mathbf{L}_+ = \mathbf{L}_1 + i\mathbf{L}_2 = -i\mathbf{P}_+$$

$$\mathbf{L}_- = \mathbf{L}_1 - i\mathbf{L}_2 = (i/2)\mathbf{K}_-$$

$$\mathbf{L}_0 = (i/2)(\mathbf{D} + \mathbf{M}_{-+})$$

$$\mathbf{E} = (i/2)(\mathbf{D} - \mathbf{M}_{-+})$$

can be traded for the algebra of differential operators acting on the field coordinates

$$[\mathbf{L}_+, \Phi(z)] \equiv L_+ \Phi(z)$$

$$[\mathbf{L}_-, \Phi(z)] \equiv L_- \Phi(z)$$

$$[\mathbf{L}_0, \Phi(z)] \equiv L_0 \Phi(z)$$

$$\begin{aligned} L_+ &= -\frac{d}{dz} \\ L_- &= \left(z^2 \frac{d}{dz} + 2jz \right) \\ L_0 &= \left(z \frac{d}{dz} + j \right) \end{aligned}$$

They satisfy the $SL(2)$ commutation relations

$$[L_0, L_{\mp}] = \pm L_{\mp}$$

$$[L_-, L_+] = 2L_0$$

The remaining generator \mathbf{E} counts the *twist* $t = \ell - s$ of the field Φ

$$[\mathbf{E}, \Phi(z)] = \frac{1}{2}(\ell - s)\Phi(z)$$

collinear twist = dimension - spin projection on the plus-direction

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Building blocks for operators:

$$q, \bar{q}, F_{\mu\nu}, D_\mu q, D_\mu D_\nu q, D_\mu D_\nu D_\rho q, \dots$$

- Tensor properties
- Equations of motion (EOM)

Baryonic twist-4 operator

$$\epsilon^{ijk} u_i^\uparrow(z_1) C \not{p} \gamma_\perp \not{p} u_j^\downarrow(z_2) \gamma^\perp \not{p} d^\downarrow(z_3)$$

Spinor Representation

Coordinates:

$$x_{\alpha\dot{\alpha}} = x_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} x_+ & w \\ \bar{w} & x_- \end{pmatrix}, \quad \sigma^{\mu} = (\mathbf{1}, \vec{\sigma})$$

To maintain Lorentz-covariance, introduce two light-like vectors $n^2 = \tilde{n}^2 = 0$

$$n_{\alpha\dot{\alpha}} = \lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}}, \quad \tilde{n}_{\alpha\dot{\alpha}} = \mu_{\alpha}\bar{\mu}_{\dot{\alpha}}$$

with auxiliary spinors λ and μ

$$x_{\alpha\dot{\alpha}} = z\lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}} + \tilde{z}\mu_{\alpha}\bar{\mu}_{\dot{\alpha}} + w\lambda_{\alpha}\bar{\mu}_{\dot{\alpha}} + \bar{w}\mu_{\alpha}\bar{\lambda}_{\dot{\alpha}}$$

Fields:

$$q = \begin{pmatrix} \psi_{\alpha} \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix}, \quad \bar{q} = (\chi^{\beta}, \bar{\psi}_{\dot{\alpha}}),$$

$$F_{\alpha\beta, \dot{\alpha}\dot{\beta}} = \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}F_{\mu\nu} = 2(\epsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta} - \epsilon_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}})$$

$f_{\alpha\beta}$ and $\bar{f}_{\dot{\alpha}\dot{\beta}}$ transform according to (1,0) and (0,1) representations of Lorentz group

“Plus” and “Minus” components

$$\begin{aligned}
 \psi_+ &= \lambda^\alpha \psi_\alpha, & \chi_+ &= \lambda^\alpha \chi_\alpha, & f_{++} &= \lambda^\alpha \lambda^\beta f_{\alpha\beta}, \\
 \bar{\psi}_+ &= \bar{\lambda}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, & \bar{\chi}_+ &= \bar{\lambda}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}, & \bar{f}_{++} &= \bar{\lambda}^{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}} \bar{f}_{\dot{\alpha}\dot{\beta}}, \\
 \psi_- &= \mu^\alpha \psi_\alpha, & \bar{\psi}_- &= \bar{\mu}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, & f_{+-} &= \lambda^\alpha \mu^\beta f_{\alpha\beta}
 \end{aligned}$$

similar for derivatives $\partial_\mu \rightarrow \partial_{\alpha\dot{\alpha}}$

$$\partial_{++} = 2\partial_z, \quad \partial_{--} = 2\partial_{\bar{z}}, \quad \partial_{+-} = 2\partial_w, \quad \partial_{-+} = 2\partial_{\bar{w}}$$

- ψ_+, χ_+, f_{++} and $\bar{\psi}_+, \bar{\chi}_+, \bar{f}_{++}$ are defined as quasipartonic

$$\psi_+ = \psi_+(z) = \psi_+(z, 0, 0, 0)$$

Operator basis for higher twists

- Operator basis containing fields and all possible derivatives is overcompleted
- In general fields with derivatives have “bad” $SL(2, \mathbb{R})$ transformation properties.
- under infinitesimal special conformal transformations in the light-cone direction: $x = \{z, \tilde{z}, w, \bar{w}\}$

$$\psi_-(x) \rightarrow \frac{1}{(1+z\epsilon)} \psi_-\left(\frac{z}{1+\epsilon z}, \tilde{z}, \frac{w}{1+\epsilon z}, \frac{\bar{w}}{1+\epsilon z}\right)$$

where from e.g.

$$[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-](z) = \frac{1}{(1+z\epsilon)^3} [D_w D_{\bar{w}} D_{\tilde{z}} \psi_-]\left(\frac{z}{1+\epsilon z}\right)$$

$\Rightarrow [D_w D_{\bar{w}} D_{\tilde{z}} \psi_-](z)$ is a “primary” field with $j = 3/2$

• but:

$$\psi_+(x) \rightarrow \frac{1}{(1+z\epsilon)^2} \left\{ \psi_+\left(\frac{z}{1+\epsilon z}, \tilde{z}, \frac{w}{1+\epsilon z}, \frac{\bar{w}}{1+\epsilon z}\right) + \epsilon z \bar{w} \psi_-(\dots) \right\}$$

\Rightarrow e.g. $[D_{\bar{w}} \psi_+](z)$ does not transform homogeneously under $SL(2, \mathbb{R})$

Solution: allow only

V. Braun, A. Manashov, J. Rohrwild, 2008

$$\psi_+(z, \tilde{z}, w, 0) = \sum_{n,k} \frac{\tilde{z}^k}{k!} \frac{w^n}{n!} [D_w^n D_{\tilde{z}}^k \psi_+](z)$$

$$\psi_-(z, \tilde{z}, 0, \bar{w}) = \sum_{n,k} \frac{\tilde{z}^k}{k!} \frac{\bar{w}^n}{n!} [D_{\bar{w}}^n D_{\tilde{z}}^k \psi_-](z)$$

and eliminate remaining “half” of transverse derivatives using EOM, e.g.

$$[D_{\bar{w}} \psi_+](z) \equiv [D_{-+} \psi_+](z) = [D_{++} \psi_-](z) + EOM = 2\partial_z \psi_-(z) + EOM$$

- similar for gluon fields

basis fields: ($E =$ collinear twist, $j =$ conformal spin)

	$j = 1/2$	$j = 1$	$j = 3/2$	$j = 2$	$j = 5/2$
$E = 1$		ψ_+			
$E = 2$	ψ_-		$D_w \psi_+$		
$E = 3$		$D_{\bar{w}} \psi_-, D_{\bar{z}} \psi_+$		$D_w^2 \psi_+$	
$E = 4$	$D_{\bar{z}} \psi_-$		$D_{\bar{w}}^2 \psi_-, D_w D_{\bar{z}} \psi_+$		$D_w^3 \psi_+$

building blocks for composite light-ray operators, e.g.

$$\mathbb{C}^{abc} \left\{ [0, z_1] \bar{\psi}_+(z_1) \right\}^a \left\{ [0, z_2] f_{++}(z_2) \right\}^b \left\{ [0, z_3] D_w \psi_+(z_3) \right\}^c$$

Premium

Manifest $SL(2)$ symmetry of higher-twist evolution equations

$SL(2)$ -invariant RG equations

$$\mathcal{O}(z_1, z_2) = \bar{\psi}_+(z_1)\psi_+(z_2)$$

Example: leading twist RG equation, $\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi}\mathbb{H}\right)[\mathcal{O}(z_1, z_2)]_R = 0$

$$\begin{aligned} [\mathbb{H} \cdot \mathcal{O}](z_1, z_2) = & 2C_F \left\{ \int_0^1 \frac{d\alpha}{\alpha} \left[2\mathcal{O}(z_1, z_2) - \bar{\alpha}\mathcal{O}(z_{12}^\alpha, z_2) - \bar{\alpha}\mathcal{O}(z_1, z_{21}^\alpha) \right] \right. \\ & \left. - \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \mathcal{O}(z_{12}^\alpha, z_{21}^\beta) - \frac{3}{2}\mathcal{O}(z_1, z_2) \right\} \end{aligned}$$

- $SL(2, R)$ -invariance

$$[\mathbb{H} \cdot L_k \mathcal{O}](z_1, z_2) = L_k[H \cdot \mathcal{O}](z_1, z_2)$$

- Two-particle representations are not degenerate

$$T^{j_1} \otimes T^{j_2} = \sum_{n=0}^{\infty} T^{j_1+j_2+n}$$

↪ \mathbb{H} can be written as a function of two-particle Casimir operator

$$\mathbb{C}_2^{SL(2,R)} = -\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} (z_1 - z_2)^2 = \widehat{J}_{12}(\widehat{J}_{12} - 1)$$

↪

Invariant representation

$$\mathbb{H} = 2C_F \left[\psi(\widehat{J}_{12} + 1) + \psi(\widehat{J}_{12} - 1) - 2\psi(1) - \frac{3}{2} \right]$$

Footnote:

to obtain this result, notice that \mathbb{H} and $\mathbb{C}_2^{SL(2,R)}$ share the same eigenfunctions:

$$\begin{aligned}\mathbb{H} \phi_n(z_1, z_2) &= h_n \phi_n(z_1, z_2) \\ \mathbb{C}_2^{SL(2,R)} \phi_n(z_1, z_2) &= j_n(j_n - 1) \phi_n(z_1, z_2)\end{aligned}$$

further, it is easy to see that

$$\phi_n(z_1, z_2) = (z_1 - z_2)^n, \quad j_n = n + 2$$

so one has to calculate action of \mathbb{H} on these polynomials and express $h_n = h(j_n)$

Footnote to the footnote:

$\phi_n(z_1, z_2) = z_{12}^n$ become Gegenbauer polynomials $C_n^{3/2}$ in adjoint representation of $SL(2)$

$$SL(2, \mathbb{R}) \rightarrow SO(4, 2)$$

Beisert, 2004, Beisert et al, 2005:

- For primary fields that we are using, the same two conditions are true with respect to the full

conformal group $SO(4, 2)$

↪ For arbitrary operators \mathbb{H} can be written as a function of $\mathbb{C}_2^{SO(4,2)}$

$$SL(2, \mathbb{R}) : \quad \mathbb{C}_2^{SL(2,R)} = J(J - 1)$$

$$SO(4, 2) : \quad \mathbb{C}_2^{SO(4,2)} = \mathbf{J}(\mathbf{J} - 1)$$

$$\mathbb{H}(J) \rightarrow \mathbb{H}(\mathbf{J})$$

the same function !

- Have to work out two-dimensional (matrix) representations:

$$\mathbb{C}_2^{SO(4,2)} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix} = \begin{pmatrix} \mathbb{C}_{++} & \mathbb{C}_{+-} \\ \mathbb{C}_{-+} & \mathbb{C}_{--} \end{pmatrix} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix}$$

$$\mathbb{H} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix} = \begin{pmatrix} \mathbb{H}_{++} & \mathbb{H}_{+-} \\ \mathbb{H}_{-+} & \mathbb{H}_{--} \end{pmatrix} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix}$$

V.M. Braun, A.N. Manashov, J. Rohrwild,

arXiv:0908.1684

Results:

$$z_{12} = z_1 - z_2$$

$$\mathbb{C}_2^{SO(4,2)} = \widehat{\mathbf{J}}(\widehat{\mathbf{J}} - 1), \quad \widehat{\mathbf{J}} = - \begin{pmatrix} 0 & \partial_2 z_{21} \\ \partial_1 z_{12} & 0 \end{pmatrix}$$

Eigenfunctions

$$\varphi_n^\pm(z_1, z_2) = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} z_{12}^n :$$

$$\mathbb{C}_2^{SO(4,2)} \varphi_n^+ = (n+2)(n+1)\varphi_n^+ \quad J = n+2$$

$$\mathbb{C}_2^{SO(4,2)} \varphi_n^- = (n+1)n\varphi_n^- \quad J = n+1$$



**Complete results for 2 → 2 RG kernels
! Not a single Feynman diagram calculated !**

Example

$$\mathcal{O}_{-+}^{ij}(z_1, z_2) = \psi_-^i(z_1)\psi_+^j(z_2), \quad \mathcal{O}_{+-}^{ij}(z_1, z_2) = \psi_+^i(z_1)\psi_-^j(z_2)$$

$$\mathbb{H} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix} = \begin{pmatrix} \mathbb{H}_{11} & \mathbb{H}_{12} \\ \mathbb{H}_{21} & \mathbb{H}_{22} \end{pmatrix} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix}$$

consider

$$\begin{aligned} \mathbb{H} \begin{pmatrix} a \\ b \end{pmatrix} z_{12}^n &= \mathbb{H} \left[\frac{a+b}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} z_{12}^n + \frac{a-b}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} z_{12}^n \right] = \frac{a+b}{2} \mathbb{H} \varphi_n^+ + \frac{a-b}{2} \mathbb{H} \varphi_n^- \\ &= \frac{a+b}{2} E(n) \varphi_n^+ + \frac{a-b}{2} E(n-1) \varphi_n^- = \begin{pmatrix} h_{11}(n) & h_{12}(n) \\ h_{21}(n) & h_{22}(n) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} z_{12}^n \end{aligned}$$

$E(n)$ is the same function as in ++ operators:

$$h_{11}(n) = \psi(n+2) + \psi(n+1) - 2\psi(1), \quad h_{12}(n) = \frac{1}{n+1}$$

obtain

$$\begin{aligned} [\mathbb{H} \mathcal{O}_{-+}^{ij}](z_1, z_2) &= -2t_{ii'}^a t_{jj'}^a \left\{ \int_0^1 \frac{d\alpha}{\alpha} \left[2\mathcal{O}_{-+}^{i'j'}(z_1, z_2) - \mathcal{O}_{-+}^{i'j'}(z_{12}^\alpha, z_2) - \bar{\alpha} \mathcal{O}_{-+}^{i'j'}(z_1, z_{21}^\alpha) \right] \right. \\ &\quad \left. + \int_0^1 d\alpha \mathcal{O}_{+-}^{i'j'}(z_{12}^\alpha, z_2) \right\} \end{aligned}$$

Does the Lorentz symmetry fix 2 \rightarrow 3 kernels?

What to do with $\mathbb{H}^{(2 \rightarrow 3)}$??

E.g. $\psi_- \psi_+, \psi_+ \psi_- \rightarrow \psi_+ \psi_+ \bar{f}_{++}$

Idea:

- Infinitesimal translation in transverse plane $P_{\mu\bar{\lambda}}$

$$i[\mathbf{P}_{\mu\bar{\lambda}}, \psi_+] = 2\partial_z \psi_- + igA_{\mu\bar{\lambda}} \psi_+ + \text{EOM},$$

- Lorentz Rotation $M_{\mu\mu}$

$$i[\mathbf{M}_{\mu\mu}, \psi_+] \sim (z\partial_z + 1)\psi_- + \frac{1}{2}igzA_{\mu\bar{\lambda}} \psi_+ + \text{EOM},$$

\hookrightarrow Exact relations between renormalized operators containing “plus” and “minus” fields \hookrightarrow The counterterms on the LHS and RHS must coincide

! It works and proves to be very efficient !

Translation

Notation

$$\begin{aligned} \mathcal{O}_{-+}^{ij}(z_1, z_2) &= \psi_{-}^i(z_1) \otimes \psi_{+}^j(z_2) & \mathcal{O}_{+-}^{ij}(z_1, z_2) &= \psi_{+}^i(z_1) \otimes \psi_{-}^j(z_2) \\ \mathcal{O}_{++}^{ij}(z_1, z_2) &= \psi_{+}^i(z_1) \otimes \psi_{+}^j(z_2) & \mathcal{O}_f^{ija}(z_1, z_2, z_3) &= \psi_{+}^i(z_1) \otimes \psi_{+}^j(z_2) \otimes \bar{f}_{++}^a(z_3) \end{aligned}$$

We are looking for three-particle counterterms

$$[\mathcal{O}_{\pm\mp}^{ij}(z_1, z_2)]'_R \sim \frac{1}{\epsilon} [\mathbb{H}_{\rightarrow f}^{(\pm\mp)} \mathcal{O}_f]^{ij}(z_1, z_2)$$

- Apply transverse derivative to leading-twist \mathcal{O}_{++} operator

$$\begin{aligned} \partial_{\mu\bar{\lambda}} [\mathcal{O}_{++}^{ij}(z_1, z_2)]'_R &= 2\partial_{z_1} [\mathcal{O}_{-+}^{ij}(z_1, z_2)]'_R + 2\partial_{z_2} [\mathcal{O}_{+-}^{ij}(z_1, z_2)]'_R \\ &\quad + ig[A_{\mu\bar{\lambda}}(z_1)\psi_{+}(z_1) \otimes \psi_{+}(z_2)]'_R \\ &\quad + ig[\psi_{+}(z_1) \otimes A_{\mu\bar{\lambda}}(z_2)\psi_{+}(z_2)]'_R + \text{EOM} \end{aligned}$$

- Convert $A_{\mu\bar{\lambda}}$ into \bar{f}_{++}

$$A_{\mu\bar{\lambda}}^b(z_1) - A_{\mu\bar{\lambda}}^b(z_2) = -z_{12}(\mu\lambda) \int_0^1 d\tau \bar{f}_{++}^b(z_{12}^\tau)$$

- rewrite the expression on the LHS:

$$\partial_{\mu\bar{\lambda}}[\mathcal{O}_{++}^{ij}(z_1, z_2)]'_R = \partial_{\mu\bar{\lambda}}\frac{1}{\epsilon}[\mathbb{H}^{++} \cdot \mathcal{O}_{++}]^{ij}(z_1, z_2)$$

→ contains two-particle and three-particle counterterms

- after a little algebra:

$$\text{LHS} \equiv \partial_1 \mathbb{H}_{\rightarrow f}^{(-+)} + \partial_2 \mathbb{H}_{\rightarrow f}^{(+ -)} = \text{RHS (known expression)}$$

- This equation is not $SL(2, R)$ invariant!

$$L_{123}^{+, (j_1 j_2 j_3)} = L_1^{+, j_1} + L_2^{+, j_2} + L_3^{+, j_3} = \sum_{k=1}^3 z_k^2 \partial_k + 2j_k z_k$$

$$(\text{LHS} - \text{RHS})L_{123}^{+, (1, 1, 3/2)} = L_{12}^{+, (1, 1)}(\text{LHS} - \text{RHS}) + (\widetilde{\text{LHS}} - \widetilde{\text{RHS}})$$

- This means that we have two equations

$$\text{LHS} = \partial_1 \mathbb{H}_{\rightarrow f}^{(-+)} + \partial_2 \mathbb{H}_{\rightarrow f}^{(+ -)} = \text{RHS},$$

$$\widetilde{\text{LHS}} = \partial_1 z_1 \mathbb{H}_{\rightarrow f}^{(-+)} + \partial_2 z_2 \mathbb{H}_{\rightarrow f}^{(+ -)} = \widetilde{\text{RHS}}$$

- Final set of equations

$$\partial_1 z_{12} \mathbb{H}_{\rightarrow f}^{(-+)} + \mathbb{H}_{\rightarrow f}^{(+ -)} = \sum_{i=1}^3 C_i \mathcal{A}_i$$

$$\partial_2 z_{21} \mathbb{H}_{\rightarrow f}^{(+ -)} + \mathbb{H}_{\rightarrow f}^{(-+)} = \sum_{i=1}^3 C_i \tilde{\mathcal{A}}_i$$

$$[\mathcal{A}_1 \varphi](z_1, z_2) = z_{12}^2 \left(\int_0^1 d\beta \bar{\beta} \varphi(z_1, z_2, z_{12}^\beta) - \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \beta \varphi(z_1, z_{21}^\alpha, z_{12}^\beta) \right)$$

$$[\mathcal{A}_2 \varphi](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta \frac{\bar{\alpha} \bar{\beta}}{\alpha} \varphi(z_1, z_{21}^\alpha, z_{12}^\beta)$$

$$[\mathcal{A}_3 \varphi](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta \frac{\bar{\alpha}}{\alpha^2} (\bar{\alpha} - \beta) \varphi(z_{12}^\alpha, z_2, z_{21}^\beta)$$

C_i are the color structures:

$$C_1 = f^{bcd} (t^b \otimes t^c), \quad C_2 = i(t^b \otimes t^d t^b), \quad C_3 = -i(t^d t^b \otimes t^b)$$

Result

$$[\mathbb{H}_{\rightarrow f}^{(-+)} \mathcal{O}_f](z_1, z_2) = z_{12}^2 \left\{ f^{abc} t^b \otimes t^c \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \beta \mathcal{O}_f(z_{12}^\alpha, z_2, z_{21}^\beta) \right. \\ \left. + i(t^a t^b) \otimes t^b \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta \frac{\bar{\alpha} \bar{\beta}}{\alpha} \mathcal{O}_f(z_{12}^\alpha, z_2, z_{21}^\beta) \right\}$$

72/(Parity × Charge Conjugation) ↦ 18 independent kernels

see [arXiv:0908.1684](https://arxiv.org/abs/0908.1684) for full list and technical details (many)

Summary

- Lorentz symmetry uniquely determines renormalization properties of operators involving higher-twist field components in terms of partonic ones
 - Efficient technique at LO (probably true beyond LO)
 - Conformal symmetry is not necessary but simplifies the analysis dramatically

We are able to show that the same results can be obtained from Lorentz symmetry alone, by applying translations and rotations to the leading-twist kernels

- Complete results for renormalization of arbitrary twist-four operators