Superfluid and turbulence

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Partition function of quantum gas with local density - density interaction:

$$\Sigma = C \int D\psi^+ D\psi e^{-S_\beta(\psi^+, \psi)},$$

ψ^+ and ψ fields obey the generalized conjugate properties $\psi^+(t, \mathbf{x}) = \psi^*(-\mathbf{t}, \mathbf{x})$, C is the normalizing constant, the fields are determined in the range $t \in [0, \beta = 1/kT]$ with the boundary conditions $\psi(\mathbf{x}, t = 0) = \psi(\mathbf{x}, t = \beta)$.

The Matsubara action S_β

$$S_\beta = \int_0^\beta dt \int d\mathbf{x} \psi^+(\mathbf{x}, t)\Big(\partial_t + \hat{H}_1\Big)\psi(\mathbf{x}, t)$$

$$+ \int_0^\beta dt \int d\mathbf{x} \int d\mathbf{x}' \psi^+(\mathbf{x}, t)\psi(\mathbf{x}, t)V(\mathbf{x} - \mathbf{x}')\psi^+(\mathbf{x}', t)\psi(\mathbf{x}', t)$$

The propagator

$$G(t', t) = e^{-\epsilon(t'-t)} \left[\Theta(t' - t) + \frac{1}{e^{\beta\epsilon} - 1} \right],$$

$$\epsilon = p^2/(2m) - \mu.$$

The propagator in the critical region

$$G(t', t) \approx \frac{1}{\beta\epsilon} = \frac{1}{\beta(p^2/(2m) - \mu)}$$

ψ, ψ^+ fields are t - independent. The action is reduced to ϕ^4 (n=2) theory.

Dynamics

t is a complex variable. An effective action is unknown, but seems to exist and to be unique.

The problem: $Re = Lv/\nu$, $\nu \rightarrow 0$, developed turbulence.

The dynamics is described by Langevin equations.

The Gibbsian action is of the form

$$S_{st} = -\partial\psi^+\partial\psi - \frac{g_1}{6}(\psi^+\psi)^2 + g_2 m\psi^+\psi - \frac{1}{2}m^2 + \frac{1}{2}v^2 + mh$$

m is a linear combination of energy and density similar to temperature fluctuations, v is a velocity field (incompressible as it is supposed in H - model and in [Antonov, Hnatic, Honkonen]).

The dynamic equations [Hohenberg, Halperin], [Васильев]

$$\partial_t \psi - \partial_i(v_i\psi) = \lambda(1 + ib)[\partial^2\psi -$$

$$-g_1(\psi^+\psi)\psi/3 + g_2 m\psi] + i\lambda g_3\psi[g_2\psi^+\psi - m + h]+$$

$$+\partial[\partial^2\psi - g_1(\psi^+\psi)\psi/3 + g_2 m\psi] + f_{\psi^+},$$

conjugated equation for ψ^+ field,

$$\partial_t m - \partial_i(v_i m) = -\lambda u\partial^2[g_2\psi^+\psi - m + h] + i\lambda g_3[\psi^+\partial^2\psi - \psi\partial^2\psi^+] + f_m,$$

$$\partial_t v = \nu\Delta v - \partial_i(v_i v) - \psi^+\partial[\partial^2\psi - \frac{g_1}{3}(\psi^+\psi)\psi + g_2 m\psi]-$$

$$-\psi\partial[\partial^2\psi^+ - \frac{g_1}{3}(\psi^+\psi)\psi^+ + g_2 m\psi^+] - m\partial[g_2\psi^+\psi - m + h] + f_v.$$

Gaussian distribution for random forces f with the correlators:

$$D_\psi(p, t, t') = \lambda\delta(t - t'), \quad D_m(p, t, t') = \lambda up^2\delta(t - t'),$$

$$D_v(p, t, t') = \nu p^2\delta(t - t').$$

These equations lead to Gibbsian limit with the static action. Let us consider the critical region $p \to 0$, $h \to 0$. Dimensional analysis:

$$d[\psi] = d/2 - 1, \quad d[m] = d/2, \quad d[\psi'] = d/2 + 1,$$

$$d[m'] = d/2, \quad d[v] = 1, \quad d[v'] = d/2,$$

the field v is IR irrelevant.

Renormalization group – ϕ^4 example

$$S = \frac{1}{2} Z_\phi^2 \partial\phi\partial\phi + \frac{g}{4!} \mu^\epsilon Z_g Z_\phi^4 \phi^4$$

$d = 4 - \epsilon$, $\gamma_i = \mu\partial_\mu \ln Z_i$, $\beta_g = g(-\epsilon - \gamma_g)$.

$$\partial_t \bar{g} = \beta(\bar{g})$$

$t \equiv \ln(k/\mu)$, $\beta(g_*) = 0$

$$\omega = \frac{\partial\beta}{\partial g}\big|_{g=g_*}$$

– matrix.

The dynamic action: F-model

$$S = 2\lambda\psi^{+\prime}\psi' - \lambda u m' \partial^2 m' + \psi^{+\prime}\{-\partial_t\psi + \lambda(1 + ib)[\partial^2\psi -$$

$$-\frac{g_1}{3}(\psi^+\psi)\psi + g_2 m\psi] + i\lambda g_3\psi[g_2\psi^+\psi - m + h]\} +$$

$$+\psi'\{-\partial_t\psi^+ + \lambda(1 - ib)[\partial^2\psi^+ -$$

$$-\frac{g_1}{3}(\psi^+\psi)\psi^+ + g_2 m\psi^+] - i\lambda g_3\psi^+[g_2\psi^+\psi - m + h]\} +$$

$$+m'\{-\partial_t m - \lambda u \partial^2[g_2\psi^+\psi - m + h] + i\lambda g_3[\psi^+\partial^2\psi - \psi\partial^2\psi^+]\}.$$

The critical behavior is described in $d = 4 - \epsilon$ dimension [De Dominicis, Peliti].

F-model: Renormalization

$$S = 2\lambda\psi^{+\prime}\psi' - \lambda um'\partial^2 m' + \psi^{+\prime}\{-\partial_t\psi + \lambda(1+ib)[\partial^2\psi-$$

$$-\frac{g_1}{3}(\psi^+\psi)\psi + g_2 m\psi] + i\lambda g_3\psi[g_2\psi^+\psi - m + h]\}+$$

$$+\psi'\{-\partial_t\psi^+ + \lambda(1-ib)[\partial^2\psi^+-$$

$$-\frac{g_1}{3}(\psi^+\psi)\psi^+ + g_2 m\psi^+] - i\lambda g_3\psi^+[g_2\psi^+\psi - m + h]\}+$$

$$+m'\{-\partial_t m - \lambda u\partial^2[g_2\psi^+\psi - m + h] + i\lambda g_3[\psi^+\partial^2\psi - \psi\partial^2\psi^+]\}.$$

1) It is multiplicative because of the Gibbsian limit.
2) In the interaction terms the differentiation can be removed at the field m', then no renormalization of $m'\partial_t m$ term and $Z'_m Z_m = 1$.
3) $\psi, \psi' \rightarrow e^{i\alpha(t)}\psi, \psi', \ \psi^+, \psi^{+\prime} \rightarrow e^{i\alpha(t)}\psi^+, \psi^{+\prime}, \ m, m' \rightarrow m, m'$ symmetry, then $Z_\lambda Z_{g_3} = Z_m$.

β - functions (for the coupling constants only)

The scaling $g_i \rightarrow g_i/(8\pi^2)$ is assumed.

$$\beta_{g_1} = g_1(-\epsilon - \gamma_{g_1}),$$
$$\beta_{g_2} = g_2(-\epsilon/2 - \gamma_{g_2}),$$
$$\beta_{g_3} = g_3(-\epsilon/2 - \gamma_{g_3}),$$
$$\beta_u = -u\gamma_u, \quad \beta_b = -b\gamma_b.$$

γ - functions (for the coupling constants only)

$$\gamma_{g_1} = -g_1\left(\frac{5}{3} - \frac{6g_2^4}{g_1^2}\right) - \frac{1}{2}g_2\left(-\frac{20}{3}g_2 + \frac{24g_2^2}{g_1}\right),$$

$$\gamma_{g_2} = -\frac{2}{3}g_1 + \frac{3}{2}g_2^2,$$

$$\gamma_{g_3} = -\frac{2b^2g_2^2u - b^2g_2^2 - 4bg_2g_3u + g_2^2u^2 - g_2^2 + 2g_3^2 + 2g_3^2u}{2((1+u)^2 + b^2)},$$

$$\gamma_u =$$

$$-\frac{b^2g_2^2u^2 - 4g_3^2b^2 - 2bg_2g_3u^2 + g_2^2u^3 - 7g_3^2u - 3g_3^2u^2 + g_2^2u^2 - 4g_3^2}{u((u+1)^2 + b^2)}$$

$$\gamma_b =$$

$$-\frac{b^2g_2^2u^2 - 4g_3^2b^2 - 2bg_2g_3u^2 + g_2^2u^3 - 7g_3^2u - 3g_3^2u^2 + g_2^2u^2 - 4g_3^2}{u((u+1)^2 + b^2)}.$$

As a result one obtain the fail of ϵ -expansion. The experimental α index: it was E model that is IR stable.

E - model: $g_2 = b = 0$

Multiplicative renormalization because of an additional symmetry $\psi \rightarrow \psi^+$, $m, m', h \rightarrow -m, -m', -h$.

Gibbsian limit, then $Z_m = 1$.

RG analysis: there are two possible critical regimes. Two-loop approximation is insufficient to choose one of them.

Viscosity critical dimension is associated with the composite operators $\nu v' \Delta v'$ and $\nu v' \Delta v$. Both produce zero contribution to the Green functions. Then higher order IR corrections are needed. It's too difficult, there are no reliable predictions for large ϵ.

The new model with the velocity correlator in the form

$$D_v = g_4 \nu^3 p^{\epsilon-\delta},$$

instead of $D_v \sim p^2$ ($d = 4 - \epsilon$).

Double δ, ϵ expansion.

Physical value of δ:

$\delta = -3$ ($\epsilon = 1$) for equilibrium model, $\delta = 4$ for developed turbulence.

The base is E - model.

The action of the new model is

$$S = 2\lambda\psi^{+\prime}\psi' - \lambda um'\partial^2 m' + v'D_v v' + \psi^{+\prime}\{-\partial_t\psi - \partial_i(v_i\psi) +$$
$$+\lambda[\partial^2\psi - g_1(\psi^+\psi)\psi/3] + i\lambda g_3\psi[-m+h]\} +$$
$$+\psi'\{-\partial_t\psi^+ - \partial_i(v_i\psi^+) + \lambda[\partial^2\psi^+ - g_1(\psi^+\psi)\psi^+/3] -$$
$$-i\lambda g_3\psi^+[-m+h]\} + m'\{-\partial_t m - \partial_i(v_i m) - \lambda u\partial^2[-m+h] +$$
$$+i\lambda g_3[\psi^+\partial^2\psi - \psi\partial^2\psi^+]\} + v'\{-\partial_t v + \nu\Delta v - \partial_i(v_i v)\}$$

The other additional terms

$$+v'\{-\psi^+\partial[\partial^2\psi - \frac{g_1}{3}(\psi^+\psi)\psi + g_2 m\psi]$$
$$-\psi\partial[\partial^2\psi^+ - \frac{g_1}{3}(\psi^+\psi)\psi^+ + g_2 m\psi^+] - m\partial[g_2\psi^+\psi - m + h]\},$$

are IR irrelevant here as in previous case.

Renormalization of the new model

1. No Gibbsian limit, non-multiplicative renormalization, new $\Delta g_3 m' \psi^+ \psi$ term is necessary.

2. Gallilei invariance, then $v\partial \sim \partial_t$.

3. ν can be presented as $\nu = u_1 \lambda$, new u_1 coupling constant.

4. There is no $v' g_4 \nu^3 v'$ counterterms, then $Z_{g_4} = Z_\nu^{-3}$. The interaction form leads to the absence of generation contributions to Z_{g_4}.

5. No $v' \partial_t v$ and $v'(v\partial v)$ counterterms as in usual hydrodynamics due to the form of interaction.

6. The renormalization constants have the form of E -model ones with additional terms proportional to g_4 and Δg_3.

One loop approximation:

1. The renormalization of v, v' part of the action is the same as in usual hydrodynamic case.

2. In the $\phi'\partial_i(v_i\phi)$ term the differentiation can be removed to ϕ' or ϕ field, then some diagramms do not contribute to renormalization constants.

Results of one-loop calculations:

(Assumed $g_4 \to g_4/(8\pi^2)$), $\qquad \gamma_{g_1} = -\frac{5}{3}g_1 - \frac{3g_4 u 1^2}{8(1 + u_1)},$

$$\gamma_{g_3} = -\frac{g_3^2}{1 + u} - g_4(-\frac{u_1^2}{8(1 + u_1)u} + \frac{3u_1^2}{8(1 + u_1)}),$$

$$\gamma_u = -g_3^2(\frac{1}{1 + u} - \frac{4}{u}) - \frac{3g_4 u_1^2}{1 + u_1}(\frac{1}{8} - \frac{1}{4u}),$$

$$\gamma_{u_1} = -\frac{g_3^2}{1 + u} - g_4(\frac{1}{8} + \frac{3u_1^2}{8(1 + u_1)}).$$

$$\gamma_\nu = \frac{g_4}{8}, \qquad \beta_{g_4} = g_4(-\delta + 3\gamma_\nu).$$

The last expression is perturbatively exact.

Two regimes for g_4 depending on signum of δ

1) Equilibrium case

$$g_{4*} = 0 \quad (\delta < 0) \quad \gamma_\nu = 0$$

2) Turbulent case

$$g_{4*} \neq 0 \quad (\delta > 0) \quad (\gamma_{\nu*} = \delta/3 > 0)$$

$$\beta_{u1} = -u_1(\gamma_\nu - \gamma_\lambda).$$

If $u_{1*} \neq 0$: $\gamma_\nu = \gamma_\lambda = \delta/3$ or 0. In last case the dynamical dimensions are determined by ω indices. If $u_{1*} = 0$: $\gamma_\nu = \delta/3$ or 0.

In equilibrium $\gamma_{\nu*} = 0$, then viscosity critical dimension is determined by ω indices (due to coupling constants mixing with g_4). Viscosity critical dimension is negative (OK).

Let's consider $\Delta g_{3*} = 0$ and $u_{1*} \neq 0$ fixed point.
Then one of the indices is $\omega = -\alpha/(2\nu)$, it is small.
The hydrodynamic index $\omega = \delta$ may be IR irrelevant.

Conclusions

1. In the equilibrium regime the viscosity critical dimension is negative. It can be determined by indices of static model.
2. The turbulence can destroy critical behaviour, viscosity is IR relevant here.
3. One- and two-loops calculations are not sufficient to solve the problem. One need the revision of phenomenological model and many-loops calculations with some resummation procedure.

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