

Quasi-oscillator presentation for the quantum group $U_q(\mathfrak{gl}_N)$

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Various presentations for $U_q(\mathfrak{gl}_N)$

Chevalley set of generators: $E_i, F_i, K_j^{\pm 1}, \quad 1 \leq i \leq N-1, \quad 1 \leq j \leq N.$

$$K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q^{\delta_{ij} - \delta_{i+1,j}} E_j, \quad [E_i, F_j] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}},$$
$$E_i^2 E_{i \pm 1} - [2]_q E_i E_{i \pm 1} E_i + E_{i \pm 1} E_i^2 = 0, \quad [E_i, E_j] = 0, \quad |i - j| \geq 2, \quad \text{etc.}$$

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Cartan-Weyl set: upper/lower triangular matrices $(L^\pm)_{ij}, \quad 1 \leq i, j \leq N.$

$$\check{R}_{12} L_1^\pm L_2^\pm = L_1^\pm L_2^\pm \check{R}_{12}, \quad \check{R}_{12} L_1^+ L_2^- = L_1^- L_2^+ \check{R}_{12},$$

where $\check{R} := \sum_{i,j=1}^N q^{\delta_{ij}} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{i>j} E_{ii} \otimes E_{jj}.$

Reflection Equation subalgebra $\mathcal{D}_N \subset U_q(\mathfrak{gl}_N)$: $L := (L^-)^{-1} L^+$.

$$\check{R}_{12} L_2 \check{R}_{12} L_2 = L_2 \check{R}_{12} L_2 \check{R}_{12}.$$

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- $\mathbb{Z}(U_q(\mathfrak{gl}_N))$ is generated by q -power sums

$$\text{Tr}_q(L^i) := \text{Tr}(DL^i), \quad \text{where } D = \text{diag}\{q^{1-2N}, \dots, q^{-3}, q^{-1}\}.$$

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- Matrix L satisfies characteristic identity

$$P(L) := L^N - q e_1 L^{N-1} + q^2 e_2 L^{N-2} + \cdots + (-q)^N e_N \mathbb{I} \equiv 0,$$

where $e_1(L) = \text{Tr}_q L$, $e_N(L) = \det_q L$, and $e_i(L)$ are related to $\text{Tr}_q(L^i)$ by a set of q -Newton relations.

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- One has a tower of RE-algebras

$$\mathcal{D}_1 \subset \cdots \subset \mathcal{D}_k \subset \cdots \subset \mathcal{D}_N$$

generated by upper-left submatrices $L^{(k)} := \|L_{ij}\|_{i,j=1}^k$, $k = 1, \dots, N$.

Spectral values

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Let's factorize characteristic polynomials for all $L^{(k)}$

$$P^{(k)}(L) := P(L^{(k)}) = \prod_{j=1}^k (L^{(k)} - \mu_j^{(k)} \mathbb{I}) \equiv 0.$$

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- Extension $\overline{\mathcal{D}}_N$ of \mathcal{D}_N with all $\frac{N(N+1)}{2}$ spectral values $\mu_j^{(k)}$, $j = 1, \dots, k$, is **not central**.
- The set $\{\mu_j^{(k)}\}_{\substack{j=1 \\ k=1}}^N$ generate a **maximal commutative subalgebra** in $\overline{\mathcal{D}}_N$.

Toy example: $U_q(\mathfrak{gl}_2)$

For 2×2 matrix $L = \begin{pmatrix} h & a^+ \\ a^- & h' \end{pmatrix}$ Reflection Equation gives

$$a^\pm h = q^{\pm 2} h a^\pm, \quad a^\pm h' = \left(h' \mp q^{\pm 1} \left(\frac{1}{q} - \frac{1}{q^3} \right) h \right) a^+,$$

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Spectral values $\mu_{1,2}$

$$q \operatorname{Tr}_q L = \frac{1}{q^2} h + h' = \mu_1 + \mu_2,$$

$$q^2 \det_q L = h \left(h' - \frac{\lambda}{q} h \right) - a^+ a^- = \frac{1}{q^2} h h' - a^- a^+ = \mu_1 \mu_2.$$

The spectral extension of $U_q(\mathfrak{gl}_2)$ is given in terms of generators

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$$a^\pm h = q^{\pm 2} h a^\pm,$$

$$a^+ a^- = -P^{(2)}(h) = -(h - \mu_1)(h - \mu_2),$$

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Why this name?

Rescale $u^+ = a^+ \frac{h^{1/2}}{q^{-2} h - \mu_2}$, $u^- = h^{-1/2} a^-$, $h = q^{2D}$ and take $\mu_1 = q^{-2}$:

$u^+ u^- = 1 - q^{2D+2}$, $u^- u^+ = 1 - q^{2D}$, $[u^\pm, D] = \pm u^\pm$ — the **q -oscillator**.

Quasi-oscillator presentation for L^\pm

$$L^+ = \begin{pmatrix} \sqrt{h} & \frac{1}{\sqrt{h}} a^+ \\ 0 & q \sqrt{\frac{\mu_1 \mu_2}{h}} \end{pmatrix}, \quad L^- = \begin{pmatrix} \frac{1}{\sqrt{h}} & 0 \\ -\frac{q}{\sqrt{h \mu_1 \mu_2}} a^- & \frac{1}{q} \sqrt{\frac{h}{\mu_1 \mu_2}} \end{pmatrix}.$$

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Additional motivation

- q -oscillators were used by V.Bazhanov, S.Lukyanov and A.Zamolodchikov for construction of the Baxter's Q -operators in their investigation of integrable structure of CFT.
- Quasi-oscillator realization of the 2×2 matrices L^\pm was later employed for construction of the Q -operators by M.Rossi, R.Weston and Ch.Korff.
- The $U_q(\mathfrak{sl}_3)$ generalizations of these constructions were considered by V.Bazhanov, A.Hibberd, S.Khoroshkin and H.Boos, F.Göhmann, A.Klümper, Kh.Nirov, A.Razumov.

Spectral extension of $U_q(\mathfrak{gl}_N)$

Generators

- $\mu_i^{(N)}$, $1 \leq i \leq N$, — N eigenvalues of L — generate the center.
- $\{a_i^{\pm(k)}, \mu_i^{(k)}\}$, $1 \leq i \leq k$, $1 \leq k \leq N-1$, — $\frac{(N-1)N}{2}$ quasi-oscillators, one for each eigenvalue of the submatrices $L^{(k)}$ — generate rest of the algebra.

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Relations among quasi-oscillators

- $a_i^{\pm(k)} \mu_j^{(p)} = q^{\pm 2\delta_{kp}\delta_{ij}} \mu_j^{(p)} a_i^{\pm(k)}$,
- $a_i^{\bullet(k)}$ and $a_j^{\bullet(p)}$ commute if $|k - p| \geq 2$ (\bullet stands for $+/-$)

Relations among quasi-oscillators

- Permutation of neighbours

$$a_i^{\pm(k)} a_j^{\mp(k+1)} = q^{\pm 1} a_j^{\mp(k+1)} a_i^{\pm(k)},$$

$$a_i^{\pm(k)} a_j^{\pm(k+1)} = \left[\frac{\mu_j^{(k+1)} - \mu_i^{(k)}}{q \mu_j^{(k+1)} - \mu_i^{(k)}/q} \right]^{\pm 1} a_j^{\pm(k+1)} a_i^{\pm(k)}.$$

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- Relations for a^\bullet 's with the same superscript

$$[a_i^{\bullet(k)}, a_j^{\bullet(k)}] = 0, \quad \text{if } i \neq j,$$

$$a_i^{+(k)} a_i^{-(k)} = -q^{2(k-1)} P^{(k-1)}(q^2 \mu_i^{(k)}) P^{(k+1)}(\mu_i^{(k)}),$$

$$a_i^{-(k)} a_i^{+(k)} = -q^{2(k-1)} P^{(k-1)}(\mu_i^{(k)}) P^{(k+1)}(\mu_i^{(k)}/q^2).$$

Expressions for L^\pm in terms of quasi-oscillators ($U_q(\mathfrak{gl}_3)$ case)

Notation: $\mu^{(1)} \mapsto h$, $\mu_i^{(2)} \mapsto \mu_i$, $\mu_i^{(3)} \mapsto \nu_i$, $a^{\pm(1)} \mapsto a^\pm$, $a^{\pm(2)}_i \mapsto b_i^\pm$.

$$L^+ = \begin{pmatrix} \sqrt{h} & \frac{1}{\sqrt{h}} a^+ & \frac{1}{q\sqrt{h}(\mu_1 - \mu_2)} \left[\frac{1}{\mu_1 - h} a^+ b_1^+ - \frac{1}{\mu_2 - h} a^+ b_2^+ \right] \\ 0 & q \sqrt{\frac{\mu_1 \mu_2}{h}} & \frac{1}{\sqrt{h \mu_1 \mu_2 (\mu_1 - \mu_2)}} [\mu_2 b_1^+ - \mu_1 b_2^+] \\ 0 & 0 & q^2 \sqrt{\frac{\nu_1 \nu_2 \nu_3}{\mu_1 \mu_2}} \end{pmatrix}$$

$$(L^-)^T = \begin{pmatrix} \frac{1}{\sqrt{h}} & -\frac{q}{\sqrt{h \mu_1 \mu_2}} a^- & -\frac{q}{\sqrt{\prod_{i,j} \mu_i \nu_j} (\mu_1 - \mu_2)} \left[\frac{\mu_2}{\mu_1 - h} b_1^- a^- - \frac{\mu_1}{\mu_2 - h} b_2^- a^- \right] \\ 0 & \frac{1}{q} \sqrt{\frac{h}{\mu_1 \mu_2}} & -\frac{1}{q \sqrt{\prod_{i,j} \mu_i \nu_j} (\mu_1 - \mu_2)} [\mu_2 b_1^- - \mu_1 b_2^-] \\ 0 & 0 & \frac{1}{q^2} \sqrt{\frac{\mu_1 \mu_2}{\nu_1 \nu_2 \nu_3}} \end{pmatrix}$$

Inverse expressions

$$b_i^+ = q\sqrt{\mu_1\mu_2} \left(\mu_i L_{21}^- L_{13}^+ + (q\mu_i - h/q) L_{11}^- L_{23}^+ \right),$$

$$b_i^- = q\mu_i \sqrt{\frac{h\nu_1\nu_2\nu_3}{\mu_1\mu_2}} \left(L_{31}^- L_{12}^+ - (\mu_i - h/q^2) L_{32}^- L_{11}^+ \right).$$

Chevalley generators in terms of quasi-oscillators

$$K_i = q^{i-1} \sqrt{\prod_j \mu_j^{(i)} / \prod_k \mu_k^{(i-1)}},$$

$$E_i = \frac{q^{i-1}}{q-1/q} \left(\prod_j \mu_j^{(i)} \right)^{-1} \sum_{j=1}^i \left(\prod_{k \neq j} \frac{\mu_k^{(i)}}{\mu_j^{(i)} - \mu_k^{(i)}} \right) a_j^{+(i)},$$

$$F_i = \frac{q^{i-1}}{q-1/q} \frac{1}{\sqrt{\prod_j \mu_j^{(i-1)} \prod_k \mu_k^{(i+1)}}} \sum_{j=1}^i \left(\prod_{k \neq j} \frac{\mu_k^{(i)}}{\mu_j^{(i)} - \mu_k^{(i)}} \right) a_j^{-(i)}.$$

q -oscillator reduction

Impose conditions

$$\mu_2^{(N)} = 1, \quad a_{i}^{\pm(k)} = 0 \quad \forall i, k \geq 2.$$

It follows that

$$\mu_i^{(k)} = q^{2(2-i)} \quad \forall i, k \geq 2.$$

and the algebra reduces to a set of $(N - 1)$ decoupled q -oscillators $\{u_k^{\pm}, D_k\}_{k=1}^{N-1}$

$$u_k^+ = \frac{\sqrt{\mu_1^{(k+1)}}}{\mu_1^{(k)} - \mu_1^{(k+1)}} \prod_{j=2}^{k-1} \frac{1}{\mu_1^{(k)} - q^{2(2-j)}} a_{1}^{+(k)},$$

$$u_k^- = \frac{1}{\sqrt{\mu_1^{(k+1)}}(\mu_1^{(k)} - \mu_1^{(k-1)})} \prod_{j=2}^k \frac{1}{\mu_1^{(k)} - q^{2(2-j)}} a_{1}^{-(k)},$$

$$q^{2D_k} = \mu_1^{(k)}.$$