Helicity and the turbulent Prandtl number

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Outline

Fully Developed Turbulence

Fully developed turbulence $\rightarrow$ Turbulence at very high Reynolds numbers, some of the possible symmetries are restored in statistical sense

- Reynolds number

$$R = \frac{LV}{\nu}, \tag{1}$$

  ν is a (kinematic) viscosity,
  L is a characteristic scale,
  V is a characteristic velocity of the flow

- Stochastic Navier-Stokes equation
  ...fluctuating part of velocity field...

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \nu \nabla^2 \mathbf{v}, && (2) \\ \nabla \cdot \mathbf{v} &= 0. && (3) \end{aligned}$$

  p is a pressure and ν is an aforementioned viscosity

Helicity

Helicity $\rightarrow$ Spatial parity violation or broken mirror symmetry

- Definition, fluid dynamics

$$H = \int \mathbf{v} \cdot (\nabla \times \mathbf{v}) d^3\mathbf{r}, \tag{4}$$

  where $\mathbf{v}$ is the velocity field
- Describes "real" world of fluid dynamics
- Example: earth's athmosphere, ocean currents
- Important mathematical model

Turbulent Prandtl Number

Turbulent Prandtl number Pr_t

- Problems of temperature diffusion (heat transfer)
- Non-dimensional parameter of turbulent flow
- Defined as ratio of the turbulent viscosity to the coefficient of the turbulent thermal diffusivity
- From experimental data: $Pr_t \in \langle 0.7, 0.9 \rangle$
- Does not depend on the individual properties of the fluid

Stochastic formulation of the model

Advection of a passive scalar field is described by the equation

$$\partial_t \phi + (\mathbf{v} \cdot \partial)\phi = u_0 \nu_0 \Delta \phi + f \tag{5}$$

u_0 is inverse Prandtl number, ν_0 is kinematical viscosity, f is a random force (explicit form of the force is not essential), $\mathbf{v}$ means incompressible velocity field (for this model)

Stochastic Navier-Stokes equation for velocity field

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \partial)\mathbf{v} = \nu_0 \Delta \mathbf{v} - \partial P + f^v \tag{6}$$

where P is a pressure and correlation function has the following form

$$\langle f_i^v(x) f_j^v(x') \rangle = \delta(t - t')(2\pi)^{-d} \int d\mathbf{k} P_{ij}(\mathbf{k}) d_f(k) \times \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}')] \tag{7}$$

d_f is some function of $k \equiv |\mathbf{k}|$ and the model parameters

Stochastic formulation of the model

Tensor characteristics of the flow are given by the term

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2 \tag{8}$$

which means incompressible isotropic velocity field
To introduce spatial parity violation or helicity into the model, we need to modificate tensor projector $P_{ij}(\mathbf{k}) \rightarrow R_{ij}(\mathbf{k})$ to the form

$$R_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2 + i \rho \, \varepsilon_{ijl} k_l / k \tag{9}$$

where ρ means helicity parameter, amount of helicity in the system. $\rho = 0$ means non helical system, $\rho = 1$ means maximum helicity in the system
ε_{ijl} is completely antisymmetric Levi-Civita's tensor of rank 3

Field theoretic formulation of the model

The stochastic model is equivalent to the quantum field model with double set of fields

$$\Phi = \{\mathbf{v}, \phi, \mathbf{v}', \phi'\} \tag{10}$$

and action functional of the model

$$\begin{aligned} S(\Phi) &= \mathbf{v}' D_f \mathbf{v}'/2 + \mathbf{v}'[-\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v}] \\ &+ \phi'[-\partial_t \phi + \nu_0 u_0 \Delta\phi - (\mathbf{v}\cdot\partial)\phi] \end{aligned} \tag{11}$$

where D_f is the correlation function of the random force. Necessary integrations over $\{t, \mathbf{x}\}$ and summations over vector indices are implied

Field theoretic formulation of the model

Propagators

$$\begin{aligned}
\langle v_i(k) v_j(k) \rangle_0 &= \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon} R_{ij}(k)}{(-i\omega + \nu_0 k^2)(i\omega + \nu_0 k^2)}, && (12)\\
\langle v_i(k) v_j'(k) \rangle_0 &= \frac{P_{ij}(k)}{-i\omega + \nu_0 k^2}, && (13)\\
\langle \phi(k) \phi'(k) \rangle_0 &= \frac{1}{-i\omega + \nu_0 u_0 k^2} && (14)
\end{aligned}$$

Vertices

$$\begin{aligned}
-\mathbf{v}'(\mathbf{v}\cdot\partial)\mathbf{v} &= v_i' \mathcal{V}_{ijs} v_j v_s / 2 \quad \textit{with} \quad \mathcal{V}_{ijs} = i(k_j \delta_{is} + k_s \delta_{ij}), && (15)\\
-\phi'(\mathbf{v}\cdot\partial)\phi &= \phi' \mathcal{V}_i v_i \phi \qquad \textit{with} \qquad \mathcal{V}_i = i k_i && (16)
\end{aligned}$$

Feynman diagrams, one-loop approximation

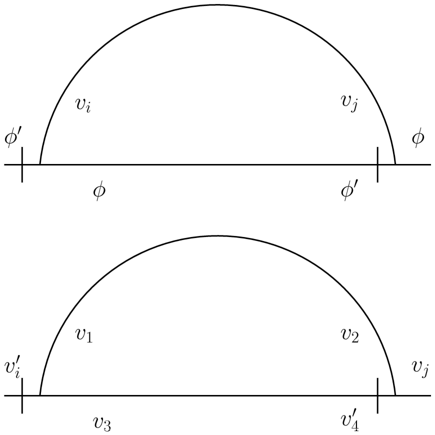
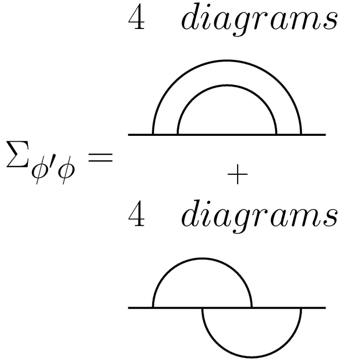


Figure: Feynman diagrams in one-loop approximation. Self energy operators $\Sigma_{\phi'\phi}$ and $\Sigma_{v'v}$.

Feynman diagrams, two-loop approximation



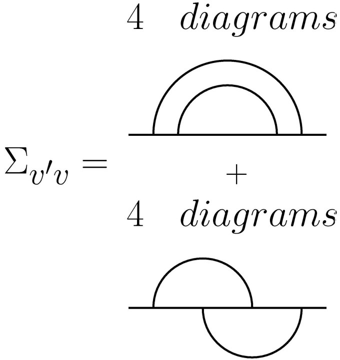


Figure: Feynman diagrams in two-loop approximation. Self energy operators $\Sigma_{\phi'\phi}$ and $\Sigma_{v'v}$.

Renormalization constants

Divergences are present only in the one-irreducible functions $\langle \phi' \phi \rangle$ and $\langle v' v \rangle$ thus we need only two independent renormalization constants
Renormalized action functional

$$\begin{aligned} S_R(\Phi) &= \mathbf{v}' D_f \mathbf{v}'/2 + \mathbf{v}'[-\partial_t \mathbf{v} + \nu Z_\nu \Delta \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v}] \\ &+ \phi'[-\partial_t \phi + \nu u Z_\kappa \Delta \phi - (\mathbf{v}\cdot\partial)\phi] \end{aligned} \tag{17}$$

By multiplicative renormalization of the parameters of the model we obtain

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad u_0 = u Z_u \tag{18}$$

with two independent renormalization constants Z_ν and Z_κ

$$Z_u = Z_\kappa Z_\nu^{-1}, \quad Z_g = Z_\nu^{-3} \tag{19}$$

Renormalization constants

Renormalization constants have the form of the Laurent expansion 1+poles in ε, (MS scheme)

$$Z = 1 + \sum_{k=1}^{\infty} a_k(g,u)\varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g_n \sum_{k=1}^{n} a_{nk}\varepsilon^{-k} \tag{20}$$

For Z_ν the following expression was obtained

$$Z_\nu = 1 + \frac{a_{11}^{(\nu)} g}{\varepsilon} + O(g^2), \quad a_{11}^{(\nu)} = -\frac{(d-1)\bar{S}_d}{8(d+2)}, \quad \bar{S}_d \equiv \frac{S_d}{(2\pi)^d} \tag{21}$$

where S_d is the area of the d-dimensional sphere

RG functions

The RG representation for response functions

$$
\begin{aligned}
\Gamma_{vv'}(k,\omega=0) &= \nu\bar{k}^2 R_v(s=1,\bar{g}), \qquad (22)\\
\Gamma_{\phi\phi'}(k,\omega=0) &= u\bar{\nu}k^2 R_\phi(s=1,\bar{g},\bar{u}), \qquad (23)\\
& \qquad (24)
\end{aligned}
$$

$\bar{g}=\bar{g}(s,g)$, $\bar{\nu}=\bar{\nu}(s,g,\nu)$ and $\bar{u}=\bar{u}(s,g,u)$ are invariant variables and satisfying RG equations of the form

$$
[-s\partial_s+\beta_g\partial_g+\beta_u\partial_u-\gamma_\nu\nu\partial_\nu]b(s,g,u)=0 \qquad (25)
$$

The RG functions are defined as follows

$$
\begin{aligned}
\beta_g &\equiv \mu\partial_\mu|_0 g=(-2\varepsilon+3\gamma_\nu), \quad \beta_u\equiv\mu\partial_\mu|_0 u=u(\gamma_\kappa-\gamma_\nu), \qquad (26)\\
\gamma_\nu(g) &\equiv \mu\partial_\mu|_0 \ln Z_\nu, \qquad \gamma_\kappa(g,u)\equiv\mu\partial_\mu|_0\ln Z_\kappa \qquad (27)
\end{aligned}
$$

Effective inverse Prandtl number

Expression for the effective inverse Prandtl number predicted by RG representation

$$u_{eff} = u_* \frac{R_\phi(s=1, g_*, u_*)}{R_v(s=1, g_*)} \tag{28}$$

The formulae for u_{eff} is universal and in the inertial range does not depend on the renormalization scheme

An iterative solution yields

$$u_* = u_*^{(1)} + u_*^{(2)}\varepsilon + \mathcal{O}(\varepsilon^2) \tag{29}$$

where

$$u_*^{(1)}[1 + u_*^{(1)}] = \frac{2(d+2)}{d} \tag{30}$$

Two-loop correction of the fixed point

$$u_*^{(2)} = \frac{2(d+2)}{d[1+2u_*^{(1)}]}\left[\lambda - \frac{128(d+2)^2}{3(d-1)^2} B(u_*^{(1)})\right] \tag{31}$$

Effective inverse Prandtl number in two-loop approximation

The two-loop approximation for the inverse turbulent Prandtl number

L. Ts. Adzhemyan, J. Honkonen, T. L. Kim and L. Sladkoff, Phys. Rev. E 71, 056311, (2005)

$$u_{eff} = u_*^{(1)}\left(1+\varepsilon\left\{\frac{1+u_*^{(1)}}{1+2u_*^{(1)}}\left[\lambda-\frac{128(d+2)^2}{3(d-1)^2}B(u_*^{(1)})\right]\right.\right.$$
$$\left.\left.+\frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}(a_v-a_\phi)\right\}\right) \quad (32)$$

where a_v and a_ϕ are integral functions of $\mathbf{k}$
λ for helicity system has the value

$$\lambda = -1.101 + 0.743\rho^2 \quad (33)$$

and numerical value of $B(u_*^{(1)})$ for $d=3$ is

$$B(u_*^{(1)}) = \sum_{i-1}^{8} b_i = -0.00443204 + 0.00382271\rho^2 \quad (34)$$

Effective inverse Prandtl number in two-loop approximation

Two-loop result for the inverse turbulent Prandtl number

$$u_{eff} = u_*^{(1)}(1 + 0.00893\varepsilon) + O(\varepsilon^2) \tag{35}$$

for non-helical system.
Behaviour of the turbulent Prandtl number on the parameter of helicity could be seen in the end of presentation

Turbulent Prandtl number

Interval of experimentally obtained values for Prandtl number

A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics: Mechanics of Trubulence, (1971)

L. P. Chua and R. A. Antonia, Int. J. Heat Mass Transfer 33, 331 (1990)

K. A. Chang and E. A. Cowen, J. Eng. Mech. 128, 1082 (2002)

$$Pr_t \in \langle 0.7, 0.9 \rangle \tag{36}$$

Two-loop value of the turbulent Prandtl number for $d = 3$

$$Pr_t = Pr_t^{(1)} + Pr_t^{(2)} = 0.7179 - 0.0128 = 0.7051 \tag{37}$$

Two-loop contribution to Pr_t is "only" about 2% and it has opposite sign with respect to the one-loop approximation.

Turbulent Prandtl number

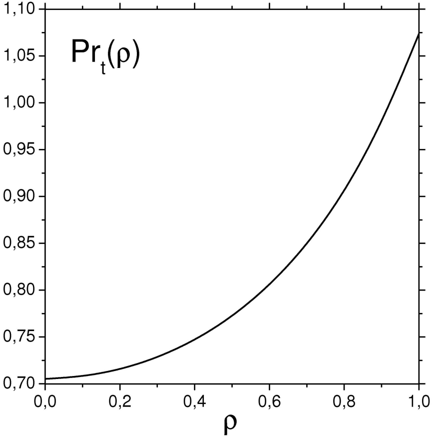


Figure: Behaviour of turbulent Prandtl number under the influence of helicity parameter ρ. Absolute value, $|\rho|$.

Conclusion

- The model of turbulent mixing of a passive scalar quantity in fully developed turbulence was studied
- The field theoretic and renormalization group approach was used
- The two-loop approximation of a corresponding expansion theory was considered
- Stable Kolmogorov regime of the model was found
- Turbulent Prandtl number and its two-loop contribution was found and compared to the one-loop approximation
- Two-loop approximation of the turbulent Prandtl number under the influence of helicity was analyzed