Antiquantization and corresponding symmetries

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Preamble

Вначале пара фраз из моей книги в качестве преамбулы. В 1971 году началась первая фаза переезда физфака в Петергоф и декан Юрий Викторович Новожилов очень помог мне и с получением квартиры и со становлением маленьким начальником – куратором курса. Юрий Викторович являл для меня пример идеального руководителя; он не докучал мелкими придирками, но в принципиальных вопросах был твердым и последовательным.

Introduction

In 1926 Ervin Schroedinger published four papers in "Annals der Physic" entitled "Quantisierung als Eigenwertprobleme". In this papers he converted nonlinear equations of classical mechanics into linear equations of quantum mechanics. Since then the latter equations are called Schroedinger equations. In 1996 I attempted to perform the inverse transform from stationary Schroedinger equation to nonlinear equation of classical mechanics in a paper published in J.Phys A [5]. This attempt was restricted from the point of view of generality but very fruitful from the point of view of classes of equations included. The class of linear equations is known as Heun class. Any equation of this class is characterized by presence of two parameters: one considered as time and the other (entitled in the theory of ODEs accessory parameter) is considered as energy. Dependance on time is an adiabatic dependance. The generated at antiquantization nonlinear equations are known as Painleve equations. They have the property that solutions are analytical functions on a Riemann suface with branching points coinsiding with singularities of the equation.

The relation between Heun equations and Painleve equations was first discussed in the paper by R. Fuchs dated 1906 [8]. This relation was studied in the frame of the so called isomonodromic condition. It needs introduction into Heun equation an additional apparent singularity.

The importance of Painleve equations follows from ARS conjecture which declares that any one-dimensional reduction of integrable equation in partial derivatives is reduced to a proper Painleve equation.

The Schroedinger equation which is in fact a second order linear equation can be converted into 2×2 linear system. Corresponding systems were also studied deeply. They are closely related to 21-st Hilbert problem which has been solved by Andrey Bolibruch [4]. These system can be converted to a system with polynomial coefficients of first degree.

Hence, integral transform can be used giving rise to numeral symmetries. Here should be mention the paper of Vladimir Fock dated 1926 where he found symmetry for Coulomb problem.

Heun class of equations

The Heun equation is a Fuchsian differential equation with four regular singularities positioned without loss of generality at $z_1 = 0$, $z_2 = 1$, $z_3 = t$, $z_4 = \infty$ in the complex z-plane. The equation itself and its various confluent cases play an important role in applied mathematics. Many known special functions of mathematical physics (hypergeometric functions, Mathieu functions, spheroidal functions, etc.) are solutions of Heun-class equations (see a review of these equations in [1]). Here, the following presentation of a Heun equation is used

$$w'' + w'\left\{\sum_{j=1}^{3} \frac{1-\theta_j}{z-z_j}\right\} + w\left\{\frac{\alpha\beta}{\sigma_3(z)} - \frac{\sigma_3(t)H}{\sigma(z)}\right\} = 0, \tag{1}$$

where $\sigma(z) = z(z-1)(z-t)$, $\sigma_j(z) = \sigma(z)/(z-z_j)$. Parameters θ_j, $j = 1, 2, 3$, are the characteristic indices for solutions with singularities at the z_j singular points whenever α, β are the characteristic indices at infinity. It should be noted that θ_3 is considered a dependent parameter because the Fuchs condition related to the characteristic indices at singularities must be satisfied (the choice of one dependent parameter θ_3 among θ_1, θ_2, and θ_3 is arbitrary). The H parameter is termed as accessory parameter of Eq. (1). It is normalized according to the condition

$$\mathrm{res}_{z=t} \frac{\sigma_3(t)}{\sigma(z)} = 1 \tag{2}$$

Hereinafter, the H parameter is termed as quantum energy. Hence, a Heun equation is characterized by 6 parameters: 4 local parameters θ_1, θ_2, α, and β, t parameter termed here as time and the H parameter.

Applying the process of confluence when singularities are merging one with the other and some coefficients tend to infinity one obtains confluent Heun equations. For instance, the biconfluent Heun equation reads

$$w''(z) - (z^2 + t)w'(z) + (\alpha z - H)W = 0$$

This equation is related to anharmonic quantum oscillator. A set of other confluent equations can be listed.

Fuchsian system

Assume that we take "rotation" matrix R as

$$R = \begin{pmatrix} z(z-1) & \rho z \\ 0 & (z-t) \end{pmatrix} \tag{3}$$

Then the inverse matrix to R is

$$R^{-1} = \sigma^{-1} \begin{pmatrix} z-t & 0 \\ -\rho z & z(z-1) \end{pmatrix}$$

Let vector $\vec{w}$ be the solution of the 2×2 system

$$\vec{w}' = \sigma^{-1} R^{-1} S \vec{w} = T \vec{w} \tag{4}$$

where $\sigma(z) = \prod_{j=1}^{3} (z - z_j)$ and

$$S = \begin{pmatrix} \alpha z + e_1 & e_2 \\ e_3 z & \beta \end{pmatrix} \tag{5}$$

Values $z_1 = 0$, $z_2 = 1$, $z_3 = t$ are locations of finite singularities of the system under consideration. Parameters α, β, e_1, e_2, e_3 obey a particular relation (Fuchs relation). This system is equivalent to Heun equation. It is only needed to find relations between parameters. Namely parameter e_3 after proper normalization corresponds to H.

Deformed Heun equations.

If an apparent singularity at the point $z_0 = q$ is added to the four regular singularities $z_1 = 0$, $z_2 = 1$, $z_3 = t$, $z_4 = \infty$ in Heun equation, we obtain a *deformed Heun equation* [2], [3] and will use the notation ***Heun1*** for such an equation. We assume that all finite singularities are real and that $0 < q < 1/2$, $t > 1$. With a proper normalization of parameters *Heun1* equation reads

$$\sigma(z)w'' + \left(\sum_{j=1}^{3}(1-\theta_j)\sigma_j(z) - \sigma(z)\frac{1}{z-q}\right)w' +$$

$$\left(\alpha\beta(z-t) - \sigma_3(t)H + \frac{\mu\sigma_3(q)(z-t)}{z-q}\right)w = 0, \quad (6)$$

where $\sigma(z) = z(z-1)(z-t)$, $\sigma_j(z) = \sigma(z)/(z-z_j)$. Whenever a Heun equation depends on 6 parameters, a *Heun1* equation (6) depends on 8 parameters with parameters q, μ added to the list. Parameters α, β, θ_j must satisfy the Fuchs condition

$$\sum_{j=1}^{3}\theta_j - \alpha - \beta = 2 \quad (7)$$

Further on we assume that

$$\theta_1 > 1 \quad (8)$$

In addition the following necessary condition (absence of logarithmic terms) holds

$$\sigma_3(t)H = \sigma(q)\mu^2 + (\sigma_3(q)+\tau(q))\mu + \alpha\beta(q-t) \quad (9)$$

where $\tau(q) = \sum_{j=1}^{3}(1-\theta_j)\sigma_j(q)$. Hence, the actual number of parameters is diminished by one while accessary parameter H can be excluded resulting in

$$\sigma(z)\left(w''\right) + \left(\sum_{j=1}^{3}(1-\theta_j)\sigma_j(z)\right)w' + \alpha\beta z w -$$

$$\left\{\sigma(q)\left(\mu^2\right) + \left(\sum_{j=1}^{3}(1-\theta_j)\sigma_j(q)\right)\mu + \alpha\beta q\right\} w$$

$$-\frac{w'-\mu w}{z-q} = 0 \qquad (10)$$

If we introduce the operator

$$D = \frac{d}{dx}$$

equation (10) can be presented in a symmetrical way

$$P(D,z) - P(\mu,q) - \frac{D-\mu}{z-q} = 0 \qquad (11)$$

The Fuchsian system equivalent to (11) can be obtained with the help of a polynomial "rotation" matrix R

$$R = \begin{pmatrix} z^2 - z & \rho(z-1) \\ z & z - t + \rho \end{pmatrix} \qquad (12)$$

The proposed choice is not unique.

Integral Euler symmetries for 2×2 linear first order system

Consider Euler integral symmetries for the following linear first order system

$$\boldsymbol{R}(z)\frac{d\boldsymbol{W}}{dz} = \boldsymbol{S}(z)\boldsymbol{W}. \tag{13}$$

Here $\boldsymbol{R}$ is already discussed polynomial "rotation" matrix of second degree and $\boldsymbol{S}$ is a linear 2×2 matrix. More precise we assume that

$$\boldsymbol{R} = \boldsymbol{A}z^2+\boldsymbol{B}z+\boldsymbol{C}, \quad \boldsymbol{S} = \kappa\boldsymbol{A}z+\boldsymbol{E}, \quad \boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution of the system (?? is sought in the form of a couture integral

$$\boldsymbol{W}(z) = \int_L (z-\zeta)^\kappa \boldsymbol{V}(\zeta)d\zeta, \tag{14}$$

where $\boldsymbol{V}(t)$ is a 2-vector function and the integration loop L is chosen under the condition that non integral terms arising at integration by parts vanish. Substituting (14) into (13) we obtain:

that $\boldsymbol{V}(\zeta)$ should be a solution of the system

$$\boldsymbol{R}(\zeta)\frac{d\boldsymbol{V}(\zeta)}{d\zeta} = [-(\kappa+2)(\boldsymbol{A}\zeta+\boldsymbol{B})+\boldsymbol{E}+\boldsymbol{B}]\,\boldsymbol{V}(\zeta), \tag{15}$$

Comparision of (13) with (15) gives symmetries for differential systems and as a result symmetries for corresponding second order equations.

Here Euler transform has been studied but for other types of equations other integral transforms can be investigated.

1 Antiquantization and Painleve equations

First comes Heun equation (1). The Painlevé equation P^6 was first derived from it in the publication by the author [5] and further presented in more detail in the book [1] and in [6]. The recipe is based on the substitution of quantum variables: coordinate, momentum and the hamiltonian for classical ones and transform from Schrödinger equation to Euler-Lagrange equation of classical movement. The hamiltonians associated with Painlevé equations are studied in books [10] and [1].

$$H(q,p,t) = \frac{1}{\sigma_3(t)}\left[\sigma(q)p^2 + \sum_{j=1}^{3}(1-\theta_j)\sigma_j(q)p + \alpha\beta(q-t)\right] \tag{16}$$

This procedure is termed as "antiquantization" and is held under condition (2). On the other hand it is not sensitive to linear transforms of independent variable and the s-homotopic transform of the dependent variable. Such an approach provides the most straightforward and simple (however, somewhat mysterious) tool for the derivation of P^6. The Legendre transform changing the variables q, p for q, $\dot{q}$ leads to the Lagrangian $L(\dot{q},\ q,t)$

$$L = \frac{\sigma_3(t)}{\sigma(q)}\left[\dot{q} - \frac{P_1(q)}{\sigma_3(t)}\right]^2 - \frac{P_2(q)}{\sigma_3(t)}$$

where

$$P_1(q) = \sum_{j=1}^{3}(1-\theta_j)\sigma_j(q), \quad P_2(q) = \alpha\beta(q-t)$$

The corresponding Euler equation.

$$\ddot{q} = \frac{1}{2}\dot{q}^2\frac{\partial \ln\sigma}{\partial q} -$$

$$\dot{q}\frac{\partial \ln(\sigma_3(t)\sigma(q))}{\partial t} +$$

$$\frac{\sigma(q)}{2\sigma_3(t)^2}\frac{\partial}{\partial q}\frac{P_1(q)^2}{\sigma(q)} +$$

$$\frac{\sigma(q)}{\sigma_3(t)}\frac{\partial}{\partial t}\frac{P_1(q)}{\sigma(q)} - \frac{2\sigma(q)}{\sigma_3(t)^2}\frac{\partial}{\partial q}P_2(q)$$

After differentiation and appropriate simplification we arrive to the following equation

$$\frac{\sigma_3(t)}{\sqrt{\sigma(q)}}\frac{d}{dt}\frac{\dot{q}\sigma_3(t)}{\sqrt{\sigma(q)}} - \dot{q}\sigma_3(t)\frac{\partial}{\partial t}\frac{1}{\sqrt{\sigma(q)}} +$$

$$\left[\frac{(2\alpha + 2 - \sum_{j=1}^{3}\theta_j)^2 + (\sum_{j=1}^{3}\theta_j)^2}{4} - \frac{t(1-\theta_1)^2}{2q^2} + \right.$$

$$\left.\frac{(t-1)(1-\theta_2)2}{2(q-1)^2} + \frac{\sigma_3(t)\theta_3^2}{(q-t)^2}\right] = 0 \quad (17)$$

This is the general form of Painlevé equation P^6 generated by Heun's equation and resulting in Kovalevskaya's dynamics.

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