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Quantization of gravitation II. Tetrad approach

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1 Introduction

Tetrad formalism (called also "frame formalism") is widely adopted for elaboration of consistent quantum theory of gravitation. Orthonormal tetrad fields are used instead of metric tensor in this formalism. Canonical approach to quantization is of greatest utility here.

In this pedagogical notes we set out the canonical method for gravitational theory in tetrad formalism. We introduce, firstly, the tetrad fields formalism in usual form and then proceed, by means of canonical transformation, to new variables, which are applicable in the so-called "loop theory of gravity", which is currently being developed.

The present notes may be regard as the sequel to "Quantization of gravitation I. Metric tensor approach" by the same authors. Cross references are denoted by letters "MT" (e. g. reference (MT10) means formula (10) of the above named notes).

2 The canonical tetrad formalism

Further we use the following notation:

$$\mu, \nu, ... = 0, 1, 2, 3, \quad A, B, ... = 0, 1, 2, 3, \quad i, k, ... = 1, 2, 3, \quad a, b, ... = 1, 2, 3, \tag{1}$$

$A, B, ..., a, b, ...$ are the tetrad indices; $\mu, \nu, ..., i, k, ...$ are coordinate indices; we take in brackets a concrete numerical value of the tetrad index (e.g. (0)), and do not do this for that of a coordinate index. In the framework of the tetrad formalism one introduces at each space-time point x^μ four mutually orthogonal[1] normalized vectors $e^\mu_A(x)$, forming a local basis in the space tangent to the space-time at that point. The index A numbers vectors, and the index μ numbers their components in usual coordinate representation.

The conditions of the orthonormalizability have the form:

$$e^\mu_A(x) g_{\mu\nu}(x) e^\nu_B(x) = \eta_{AB}, \tag{2}$$

where the η_{AB} is Lorentz metric tensor;

$$\eta_{AB} = diag(-1, 1, 1, 1). \tag{3}$$

It is assumed that the vectors $e^\mu_A(x)$ are linearly independent at each point (x), i. e. that

$$\det\left(e^\mu_A(x)\right) \neq 0. \tag{4}$$

Therefore it is possible to introduce the quantities $e^\mu_A(x)$, taking

$$e^A_\mu(x) e^\nu_A(x) = \delta^\nu_\mu, \tag{5}$$

and hence,

$$e^A_\mu(x) e^\mu_B(x) = \delta^A_B. \tag{6}$$

According to (2), (5),

$$g_{\mu\nu}(x) = e^A_\mu(x) \eta_{AB} e^B_\nu(x). \tag{7}$$

[1] We use here and in the following the term "orthogonal" in the sense of pseudoriemannian fourdimensional metric.

The set of the four vectors introduced at each point is called pseudo-orthogonal tetrad (or vierbein), and the quantities e^{μ}_{A} and e^{A}_{μ} are called frame parameters or tetrads. Applying the tetrad formalism, one considers frame parameters $e^{A}_{\mu}(x)$ as dynamical variables, describing the gravitational field, while the metric $g_{\mu\nu}(x)$ is to be a function of those variables (according to (7)).

To define the theory the expression (7) for the $g_{\mu\nu}$ is substituted into the action of the gravitational field. Varying the obtained action w. r. t. the $e^{A}_{\mu}(x)$, one gets the field equations, equivalent to Einstein equations. We get the theory, invariant under two groups of local transformations: the group of coordinate transformations, at which the vectors, referred to coordinate basis, transform in usual way

$$a'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu}(x), \qquad a'_{\mu}(x') = a_{\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}}, \tag{8}$$

while the frame parameters transform under the rule

$$e'^{\mu}_{A}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} e^{\nu}_{A}(x), \qquad e'^{A}_{\mu}(x') = e^{A}_{\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}}, \tag{9}$$

and the group of local Lorentz transformations, at which the vectors, referred to the tetrad basis, change under the formulas

$$a'^{A}(x) = \omega^{A}{}_{B}(x) a^{B}(x), \qquad a'_{A}(x) = a_{B}(x)\omega^{-1B}{}_{A}(x) \tag{10}$$

and the frame parameters transform in accord with the equalities

$$e'^{A}_{\mu}(x) = \omega^{A}{}_{B}(x) e^{B}_{\mu}(x), \tag{11}$$

$$e'^{\mu}_{A}(x) = e^{\mu}_{B}(x)\omega^{-1B}{}_{A}(x) a^{B}(x). \tag{12}$$

Here the $\omega^{A}{}_{B}(x)$ is a matrix of Lorentz transformation, i. e. such a matrix that

$$\eta_{AB}\omega^{A}{}_{D}(x)\omega^{B}{}_{E}(x) = \eta_{DE}. \tag{13}$$

That's why in accord with the (7), the metric $g_{\mu\nu}$ does not change under Lorentz transformations (11), (12).

Using the parameters e^{A}_{μ}, e^{μ}_{A} , it is possible to transform the vectors, referred to the coordinates basis, into the vectors, referred to tetrad basis:

$$a^{A} = e^{A}_{\mu} a^{\mu}, \quad a_{A} = e^{\mu}_{A} a_{\mu}, \qquad a^{\mu} = e^{\mu}_{A} a^{A}, \quad a_{A} = e^{A}_{\mu} a_{A}. \tag{14}$$

Vectors, connected by such relations, are considered as different representations of the same vector.

It is also possible to define tensors with the indices, referred to coordinates or tetrad basis. Such a tensor $T^{A\ldots\alpha\ldots}_{B\ldots\beta\ldots}$ transforms w. r. t. every index like the corresponding vector. With the help of the parameters e^{A}_{μ}, e^{μ}_{A} it is possible to change the indices of tensors (like of the vectors: A to μ and μ to A).

Analogously to covariant derivatives, referred to coordinate basis,

$$\nabla_{\mu} a^{\alpha} = \partial_{\mu} a^{\alpha} + \Gamma^{\alpha}_{\mu\beta} a^{\beta}, \tag{15}$$

$$\nabla_{\mu} a_{\alpha} = \partial_{\mu} a_{\alpha} - a_{\beta}\Gamma^{\beta}_{\mu\alpha}, \tag{16}$$

one can introduce the covariant derivatives of the vectors, referred to the tetrad basis:

$$\nabla_\mu a^A = \partial_\mu a^A + A_\mu{}^A{}_B a^B, \qquad \nabla_\mu a_A = \partial_\mu a_A - a_B A_\mu{}^B{}_A. \tag{17}$$

Accordingly the covariant derivative of the tensor is defined by the rule

$$\nabla_\mu T^{A\ldots\alpha\ldots}_{B\ldots\beta\ldots} = \partial_\mu T^{A\ldots\alpha\ldots}_{B\ldots\beta\ldots} + A_\mu{}^A{}_D T^{D\ldots\alpha\ldots}_{B\ldots\beta\ldots} + \ldots + \\ +\Gamma^\alpha_{\mu\delta} T^{A\ldots\delta\ldots}_{B\ldots\beta\ldots} - \ldots - T^{A\ldots\alpha\ldots}_{D\ldots\beta\ldots} A_\mu{}^D{}_B - \ldots - T^{A\ldots\alpha\ldots}_{B\ldots\gamma\ldots}\Gamma^\gamma_{\mu\beta} - \ldots. \tag{18}$$

The connection $A_\mu{}^A{}_B$ is chosen to be determined by the following relation:

$$\nabla_\mu \left(e^A_\nu a^\nu\right) = e^A_\nu \nabla_\mu a^\nu, \tag{19}$$

therefore

$$\nabla_\mu e^A_\nu = 0, \tag{20}$$

i. e.

$$\partial_\mu e^A_\nu + A_\mu{}^A{}_B e^B_\nu - e^A_\lambda \Gamma^\lambda_{\mu\nu} = 0, \tag{21}$$

hence,

$$A_\mu{}^A{}_B = e^A_\lambda \Gamma^\lambda_{\mu\nu} e^\nu_B - \left(\partial_\mu e^A_\nu\right) e^\nu_B = e^A_\lambda \Gamma^\lambda_{\mu\nu} e^\nu_B + e^A_\nu \partial_\mu e^\nu_B \tag{22}$$

and

$$\Gamma^\lambda_{\mu\nu} = e^\lambda_A A_\mu{}^A{}_B e^B_\nu + e^\lambda_A \partial_\mu e^A_\nu. \tag{23}$$

From the usual expression for the $\Gamma^\lambda_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = -\frac{1}{2} g^{\lambda\varphi}(\partial_\mu g_{\varphi\nu} + \partial_\nu g_{\varphi\mu} - \partial_\varphi g_{\mu\nu}) \tag{24}$$

and the equalities (7), (22), (23) it is possible to derive the relation

$$A_{\mu,AB} \equiv \eta_{AD} A_\mu{}^D{}_B = -\eta_{AD} S^D_{\mu\alpha} e^\alpha_B + \eta_{BD} S^D_{\mu\alpha} e^\alpha_A - e^E_\mu \eta_{ED} e^\alpha_A S^D_{\alpha\beta} e^\beta_B, \tag{25}$$

where

$$S^D_{\alpha\beta} \equiv \frac{1}{2}\left(\partial_\alpha e^D_\beta - \partial_\beta e^D_\alpha\right). \tag{26}$$

The transformation law of the $A_\mu{}^A{}_B$ under the frame transformations is found from the requirement

$$\nabla'_\mu a'^A \equiv \nabla'_\mu(\omega^A{}_B a^B) = \omega^A{}_B \nabla_\mu a^B \tag{27}$$

and has the form

$$A'_\mu{}^A{}_B = \omega^A{}_D A_\mu{}^D{}_E (\omega^{-1})^E{}_B + \omega^A{}_D \partial_\mu (\omega^{-1})^D{}_B. \tag{28}$$

As follows from (25)

$$A_{\mu,AB} = -A_{\mu,BA}, \tag{29}$$

in virtue of that the $A_{\mu,AB}$ can be decomposed in generators of Lorentz group:

$$A_{\mu}^{A}{}_{B}(x) = A_{\mu}^{a}(x) T_{a}{}^{A}{}_{B}, \tag{30}$$

where $a = 1, 2, ..., 6$, and the $T_a{}^A{}_B$ do not depend on the x^μ.

The complete group of the symmetry has 10 local parameters (the 4 functions of the coordinate transformation and the 6 parameters of Lorentz transformation). Therefore it is necessary to introduce 10 extra conditions, fixing this arbitrariness. Of course, it is possible to remove at first only a part of the arbitrariness. We will use this in the construction of the canonical formalism for that theory. We restrict partly the choice of tetrads, connecting the tetrads with the coordinate system by the condition

$$e^{\mu}_{(0)} = n^{\mu}, \tag{31}$$

where the n^μ is a normal to the surface $x^0 = const$ at the given point. Because $n_i = 0$ it follows from (31) that

$$e^{0}_{a} n_0 = e^{\mu}_{a} n_\mu = e^{\mu}_{a} g_{\mu\nu} n^\nu = e^{\mu}_{a} g_{\mu\nu} e^{\nu}_{(0)} = \eta_{a(0)} = 0. \tag{32}$$

Therefore

$$e^{0}_{a} = 0. \tag{33}$$

In fixed coordinate system one has a freedom to perform local $O(3)$ – transformations of tetrads. Under the change of coordinates the tetrads also change so that the conditions (31), (33) always hold. At the given change of coordinates the change of the tetrads is defined up to $O(3)$ – transformations. The remaining group of all transformations is a semi-direct product of coordinate group onto $O(3)$ – tetrad group. Since the conditions (31), (33) do not restrict the metric $g_{\mu\nu}$, these conditions do not violate the equivalence of classical theory in terms of frame parameters and the theory in terms of metric tensor.

According to (MT86), (MT87)

$$N_i = -\beta_{ik}\frac{n^k}{n^0}, \quad N = \frac{1}{n^0}, \tag{34}$$

hence, (3-dimensional coordinate indices are lifted up and pull down with the help of β^{ik} and β_{ik})

$$N^k = -\frac{n^k}{n^0}, \quad n^0 = \frac{1}{N}, \quad n^k = -n^0 N^k, \tag{35}$$

$$n^0 = \frac{1}{N}, \quad n^k = -\frac{N^k}{N}. \tag{36}$$

Thus, in accord with the (31), (33), (36)

$$e^{0}_{(0)} = \frac{1}{N}, \quad e^{0}_{a} = 0, \quad e^{i}_{(0)} = -\frac{N^i}{N}, \tag{37}$$

e^{i}_{a} are not expressed through the N and N^i. The quantities e^{A}_{ν} are defined by the condition

$$e^{\mu}_{A} e^{A}_{\nu} = \delta^{\mu}_{\nu} \tag{38}$$

or

$$e^0_{(0)}e^{(0)}_0 + e^0_a e^a_0 = 1, \tag{39}$$

$$e^0_{(0)}e^{(0)}_i + e^0_a e^a_i = 0, \tag{40}$$

$$e^i_{(0)}e^{(0)}_0 + e^i_a e^a_0 = 0, \tag{41}$$

$$e^i_{(0)}e^{(0)}_k + e^i_a e^a_k = \delta^i_k. \tag{42}$$

According the (37) $e^0_a = 0$. Therefore eq-ns (39) and (40) take the form

$$e^0_{(0)}e^{(0)}_0 = 1, \tag{43}$$

$$e^0_{(0)}e^{(0)}_i = 0. \tag{44}$$

In accord with the (37) $e^0_{(0)} = \frac{1}{N}$. So, one gets from (43), (44)

$$e^{(0)}_0 = N, \quad e^{(0)}_i = 0. \tag{45}$$

Consequently (42) takes the form

$$e^i_a e^a_k = \delta^i_k \tag{46}$$

and the (41) becams

$$e^i_{(0)}N + e^i_a e^a_0 = 0. \tag{47}$$

According to the (37) $e^i_{(0)} = -\frac{N^i}{N}$. Therefore, the (47) takes the form

$$e^i_a e^a_0 = N^i \tag{48}$$

and owing to (46)

$$e^a_0 = e^a_i N^i. \tag{49}$$

The conditions (46) and (6), i. e.

$$e^\mu_A e^A_\nu = \delta^\mu_\nu \tag{50}$$

are fulfilled simultaneously. Denote

$$e \equiv \det(e^a_i). \tag{51}$$

We have

$$g_{\mu\nu} = e^A_\mu \eta_{AB} e^B_\nu = -e^{(0)}_\mu e^{(0)}_\nu + e^a_\mu e^a_\nu. \tag{52}$$

According to the (45) $e^{(0)}_i = 0$. So, it follows from the (52) that

$$\beta_{ik} = g_{ik} = e^a_i e^a_k. \tag{53}$$

Hence, according to (37),

$$g^{ik} = -e^i_{(0)} e^k_{(0)} + e^i_a e^k_a = e^i_a e^k_a - \frac{N^i N^k}{N^2}. \tag{54}$$

In agreement with the (MT100) and (54)

$$\beta^{ik} = g^{ik} + \frac{N^i N^k}{N^2} = e^i_a e^k_a. \tag{55}$$

Eq-ns (53), (55) are in accord with the (46). The quantities g_{00}, g_{0i}, g^{00}, g^{0i} are expressed in terms of N, N^i, β_{ik} by formulas (MT100)-(MT102).

Thus, all components of the 4-dimensional metric are expressed in terms of e^a_i (or e^i_a) and N, N^i, while no restrictions on the components of the metric arise. Owing to (53), (51)

$$\beta \equiv \det(\beta_{ik}) = e^2, \quad e = \sqrt{\beta}. \tag{56}$$

Let us introduce

$$Q^i_a = \sqrt{\beta} e^i_a = e e^i_a \tag{57}$$

and define Q^a_i by the equality

$$Q^a_i Q^k_a = \delta^k_i. \tag{58}$$

Denote

$$Q \equiv \det(Q^i_a). \tag{59}$$

Then, owing to (57)

$$Q = e^3 e^{-1} = e^2 = \beta, \quad e = \sqrt{Q}. \tag{60}$$

According to (57), (58), (60), (46)

$$Q^a_i = e^{-1} e^a_i = \frac{1}{\sqrt{\beta}} e^a_i. \tag{61}$$

Conversely, from (57), (61)

$$e^i_a = Q^{-\frac{1}{2}} Q^i_a, \tag{62}$$

$$e^a_i = Q^{\frac{1}{2}} Q^a_i. \tag{63}$$

In accord with the (53), (55), (62), (63), (60)

$$\beta_{ik} = Q Q^a_i Q^a_k, \tag{64}$$

$$\beta^{ik} = Q^{-1} Q^i_a Q^k_a = \beta^{-1} Q^i_a Q^k_a. \tag{65}$$

In agreement with the (MT355), (65) the Faddeev-Popov (FP) variables q^{ik} are expressed simply through the Q^i_a:

$$q^{ik} = \beta \beta^{ik} = Q^i_a Q^k_a. \tag{66}$$

For the further applications we choose as initial variables in the tetrad formalism the quantities: Q^i_a, N, N^i. Indices $a, b, ...$ are lifted up and sinking down with the help of the tensors $\eta_{ab} = \delta_{ab}$,$\eta^{ab} = \delta^{ab}$. Therefore it is nonessential where one writes the frame indices $a, b, \ldots$, up or down. Further we do not pay attention to where these indices are placed, and write

$$Q^a_i \equiv Q_{ia}, \quad Q^i_a \equiv Q^{ia}. \tag{67}$$

However, this does not concern the indices $i, k, ...$, which are lowered and lifted with the β_{ik}, β^{ik}.

Evidently,

$$e^a_i \beta^{ik} = e^a_i e^i_b e^k_b = \delta^a_b e^k_b = e^k_a \equiv e^{ka}. \tag{68}$$

But one must have in mind that owing to (68), (57), (61)

$$e^2 e^{-1} e^a_i \beta^{ik} = e e^k_a, \tag{69}$$

i. e.

$$e^2 Q^a_i \beta^{ik} = Q^k_a, \tag{70}$$

so that

$$Q^k_a \neq \beta^{ki} Q^a_i, \tag{71}$$

and by (60)

$$Q^k_a = e^2 \beta^{ki} Q^a_i = Q \beta^{ki} Q^a_i. \tag{72}$$

The first-order Lagrangian in tetrad formalism can be obtained in the simplest way from the 1st-order Lagrangian in the FP variables (MT372)

$$\mathcal{L}_{(1)} = \pi_{ik} \partial_0 q^{ik} + (\text{terms without } \partial_0(\ldots)) \,. \tag{73}$$

By (73) and (66)

$$\mathcal{L}_{(1)} = \pi_{ik} \partial_0 (Q^i_a Q^k_a) + (\text{terms without } \partial_0(\ldots)) = \\ = 2\pi_{ik} Q^k_a \partial_0 Q^i_a + (\text{terms without } \partial_0(\ldots)) \,. \tag{74}$$

We find from this the momenta $\mathcal{P}^a_i$, conjugated with the Q^i_a:

$$\mathcal{P}^a_i = \frac{\partial \mathcal{L}_{(1)}}{\partial(\partial_0 Q^i_a)} = 2\pi_{ik} Q^k_a. \tag{75}$$

Owing to that

$$\pi_{ik} = \frac{1}{2} Q^a_k \mathcal{P}^a_i. \tag{76}$$

Substituting the expressions (76) and $q^{ik} = Q^i_a Q^k_a$ into the $\mathcal{L}_{(1)}$ from (MT372), we obtain the 1st-order Lagrangian of tetrad formalism. However, it is necessary to take into account that from the (75) three new constraints arise. Since $\pi_{ik} = \pi_{ki}$, we have by (76)

$$Q^a_k \mathcal{P}^a_i - Q^a_i \mathcal{P}^a_k = 0. \tag{77}$$

These relations are equivalent to the equalities

$$Q^k_b Q^i_c \left(Q^a_k \mathcal{P}^a_i - Q^a_i \mathcal{P}^a_k\right) = 0, \tag{78}$$

or

$$Q^{ic}\mathcal{P}^b_i - Q^{ib}\mathcal{P}^c_i = 0, \tag{79}$$

i. e.

$$\Phi^a \equiv \varepsilon^{abc} Q^{ib}\mathcal{P}^c_i = 0. \tag{80}$$

Let us show that the constraints Φ^a are the generators of the $O(3)$-rotations of tetrads around the fixed directions $e^{\mu}_{(0)}$ (in a sense of Poisson brackets). Here and later we adopt notations (if not whatever specified) $x^0 = \tilde{x}^0 = 0$, $x = (0,\, x^1,\, x^2,\, x^3)$, $\tilde{x} = (0,\, \tilde{x}^1,\, \tilde{x}^2,\, \tilde{x}^3)$, $\delta^3(x-\tilde{x}) = \delta(x^1-\tilde{x}^1)\delta(x^2-\tilde{x}^2)\delta(x^3-\tilde{x}^3)$ as well as $Q^i_a(x) \equiv Q^i_a$, $Q^i_a(\tilde{x}) \equiv \underset{\sim}{Q}{}^i_a$ and similarly for other functions. Let $\xi^a(x) \equiv \xi^a(x^1, x^2, x^3)$ is an infinitesimal function. Then, at $x^0 = const$

$$\left\{\int d^3x \xi^a(x)\Phi^a(x), Q^{ld}(\widetilde{x})\right\} = \int d^3x \xi^a \varepsilon^{abc} Q^{kb}\left\{\mathcal{P}^c_k, \underset{\sim}{Q}{}^{ld}\right\} =$$
$$= \int d^3x \xi^a \varepsilon^{abc} Q^{kb}\left(-\delta^{cd}\delta^l_k \delta^3(x-\widetilde{x})\right) = -\underset{\sim}{\xi}{}^a \varepsilon^{abd} \underset{\sim}{Q}{}^{lb} = \varepsilon^{dba} \underset{\sim}{Q}{}^{lb} \underset{\sim}{\xi}{}^a. \tag{81}$$

Here we have taken into account that

$$\left\{Q^i_a, \underset{\sim}{\mathcal{P}}{}^b_k\right\} = \delta^b_a \delta^i_k \delta^3(x-\widetilde{x}), \tag{82}$$

$$\left\{Q^i_a, \underset{\sim}{Q}{}^k_b\right\} = \left\{\mathcal{P}^a_i, \underset{\sim}{\mathcal{P}}{}^b_k\right\} = 0. \tag{83}$$

Analogously,

$$\left\{\int d^3x\, \xi^a \Phi^a, \underset{\sim}{\mathcal{P}}{}^d_l\right\} = \int d^3x\, \xi^a \varepsilon^{abc}\left\{Q^{kb}, \underset{\sim}{\mathcal{P}}{}^d_l\right\}\mathcal{P}^c_k =$$
$$= \int d^3x\, \xi^a \varepsilon^{abc} \delta^{bd} \delta^k_l \delta^3(x-\widetilde{x})\mathcal{P}^c_k = \varepsilon^{adc} \underset{\sim}{\xi}{}^a \underset{\sim}{\mathcal{P}}{}^c_l = \varepsilon^{dba} \underset{\sim}{\mathcal{P}}{}^b_l \underset{\sim}{\xi}{}^a. \tag{84}$$

The equalities (81) and (84) mean that the Φ^a are generators of the $O(3)$ – transformations of tetrads. Therefore the Φ^a commute (in a sense of Poisson brackets) with all the quantities, invariant w. r. t. such transformations and composed only from Q^i_a and $\mathcal{P}^a_i$. In particular, if the constraints $\mathcal{H}_0$ and $\mathcal{H}_i$ are expressed in terms of Q^i_a and $\mathcal{P}^a_i$, we have

$$\left\{\Phi^a, \underset{\sim}{\mathcal{H}}{}_0\right\} = 0, \tag{85}$$

$$\left\{\Phi^a, \underset{\sim}{\mathcal{H}}{}_i\right\} = 0. \tag{86}$$

Since, the Φ^a as well as Q^{ia} is a frame $O(3)$-vector, we get, analogously to (81)

$$\left\{\int d^3x\, \xi^a \Phi^a, \underset{\sim}{\Phi}{}^b\right\} = \varepsilon^{bca} \underset{\sim}{\Phi}{}^c \underset{\sim}{\xi}{}^a \tag{87}$$

or

$$\int d^3x\, \xi^a \left(\left\{\Phi^a, \underset{\sim}{\Phi}^b\right\} - \varepsilon^{bca}\Phi^c\delta(x-\widetilde{x})\right) = 0. \tag{88}$$

Because the $\xi^a(x)$ is an arbitrary function we have

$$\left\{\Phi^a, \underset{\sim}{\Phi}^b\right\} = \varepsilon^{abc}\Phi^c\delta(x-\widetilde{x}). \tag{89}$$

Thus, the commutators of the Φ^a with all other constrains are again constrains, and therefor the Φ^a are first class constrains. At last, by (66), (76), (83), (82) we get after the q^{ik} and π_{lm} being expressed through the Q^i_a, $\mathcal{P}^a_i$:

$$\left\{q^{ik}, \underset{\sim}{q}^{lm}\right\} = \left\{Q^i_a Q^k_a, \underset{\sim}{Q}{}^l_b \underset{\sim}{Q}{}^m_b\right\} = 0, \tag{90}$$

$$\begin{aligned}
\left\{\pi_{ik}, \underset{\sim}{\pi}_{lm}\right\} &= \frac{1}{4}\left\{Q^a_k\mathcal{P}^a_i, \underset{\sim}{Q}{}^b_m\underset{\sim}{\mathcal{P}}{}^b_l\right\} = \frac{1}{4}\left\{Q^a_k, \underset{\sim}{Q}{}^b_m\mathcal{P}^b_l\right\}\mathcal{P}^a_i + \frac{1}{4}Q^a_k\left\{\mathcal{P}^a_i, \underset{\sim}{Q}{}^b_m\underset{\sim}{\mathcal{P}}{}^b_l\right\} = \\
&= \frac{1}{4}Q^b_m\left\{Q^a_k, \underset{\sim}{\mathcal{P}}{}^b_l\right\}\mathcal{P}^a_i + \frac{1}{4}Q^a_k\left\{\mathcal{P}^a_i, \underset{\sim}{Q}{}^b_m\right\}\underset{\sim}{\mathcal{P}}{}^b_l = \\
&= \frac{1}{4}\underset{\sim}{Q}{}^b_m(-Q^a_rQ^c_k\left\{Q^r_c, \underset{\sim}{\mathcal{P}}{}^b_l\right\})\mathcal{P}^a_i + \frac{1}{4}Q^a_k(-\left\{\mathcal{P}^a_i, \underset{\sim}{Q}{}^r_c\right\}\underset{\sim}{Q}{}^b_r\underset{\sim}{Q}{}^c_m)\underset{\sim}{\mathcal{P}}{}^b_l = \\
&= -\frac{1}{4}\underset{\sim}{Q}{}^b_m(Q^a_rQ^c_k\delta^r_l\delta^b_c\delta^3(x-\widetilde{x}))\mathcal{P}^a_i - \frac{1}{4}Q^a_k(-\delta^r_i\delta^a_c\delta^3(x-\widetilde{x})\underset{\sim}{Q}{}^b_rQ^c_m)\underset{\sim}{\mathcal{P}}{}^b_l = \\
&= -\frac{1}{4}Q^b_mQ^a_lQ^b_k\delta^3(x-\widetilde{x})\mathcal{P}^a_i + \frac{1}{4}Q^a_kQ^b_iQ^a_m\delta^3(x-\widetilde{x})\mathcal{P}^b_l = -\frac{1}{4}Q^b_mQ^b_k(Q^a_l\mathcal{P}^a_i - Q^a_i\mathcal{P}^a_l)\delta^3(x-\widetilde{x}) = \\
&= \frac{1}{4}Q^b_mQ^b_kQ^d_iQ^c_l(Q^{nc}\mathcal{P}^d_n - Q^{nd}\mathcal{P}^c_n)\delta^3(x-\widetilde{x}) = \\
&= \frac{1}{4}Q^b_mQ^b_kQ^d_iQ^c_l\varepsilon^{cda}\varepsilon^{aef}Q^{ne}\mathcal{P}^f_n\delta^3(x-\widetilde{x}) = \frac{1}{4}Q^b_mQ^b_kQ^d_iQ^c_l\varepsilon^{cda}\Phi^a\delta^3(x-\widetilde{x}),
\end{aligned} \tag{91}$$

i. e.

$$\{\pi_{ik}, \pi_{lm}\} = \frac{1}{4}Q^b_mQ^b_kQ^d_iQ^c_l\varepsilon^{cda}\Phi^a\delta^3(x-\widetilde{x}). \tag{92}$$

Further,

$$\begin{aligned}
\left\{q^{ik}, \underset{\sim}{\pi}_{lm}\right\} &= \frac{1}{2}\left\{Q^i_aQ^k_a, \underset{\sim}{Q}{}^b_m\underset{\sim}{\mathcal{P}}{}^b_l\right\} = \frac{1}{2}Q^b_m\left\{Q^i_aQ^k_a, \underset{\sim}{\mathcal{P}}{}^b_l\right\} = \\
&= \frac{1}{2}\underset{\sim}{Q}{}^b_m\left\{Q^i_a, \underset{\sim}{\mathcal{P}}{}^b_l\right\}Q^k_a + \frac{1}{2}\underset{\sim}{Q}{}^b_mQ^i_a\left\{Q^k_a, \underset{\sim}{\mathcal{P}}{}^b_l\right\} = \frac{1}{2}\underset{\sim}{Q}{}^b_m\delta^i_l\delta^b_a\delta^3(x-\widetilde{x})Q^k_a + \frac{1}{2}\underset{\sim}{Q}{}^b_mQ^i_a\delta^k_l\delta^a_b\delta^3(x-\widetilde{x}) = \\
&= \frac{1}{2}Q^b_mQ^k_b\delta^i_l\delta^3(x-\widetilde{x}) + \frac{1}{2}Q^b_mQ^i_b\delta^k_l\delta^3(x-\widetilde{x}) = \frac{1}{2}(\delta^k_m\delta^i_l + \delta^i_m\delta^k_l)\delta^3(x-\widetilde{x}),
\end{aligned} \tag{93}$$

i. e.

$$\left\{q^{ik}, \underset{\sim}{\pi}_{lm}\right\} = \delta^{ik}_{lm}\delta^3(x-\widetilde{x}). \tag{94}$$

Thus, expressing q^{ik} and π_{lm} through the Q^i_a and $\mathcal{P}^b_l$ by (66), (76), we find the relations (90), (92), (94). If we substitute now the expressions for the q^{ik} and π_{lm} in terms of Q^i_a

and $\mathcal{P}_l^b$ into the constraints $\mathcal{H}_0$ and $\mathcal{H}_i$, then, after commutation of these constraints, the same expressions as in the FP-formalism arise, up to the terms, proportional to the Φ^a, because of the change of the commutator (92). Taking into account the said above and the formulas (MT328)-(MT330), we get the algebra of the constraints:

$$\left\{\mathcal{H}_i, \underset{\sim}{\mathcal{H}_k}\right\} = \mathcal{H}_k\partial_i\delta^3(x-\widetilde{x}) + \underset{\sim}{\mathcal{H}_i}\underset{\sim}{\partial_k}\delta^3(x-\widetilde{x}) + (...)^a\Phi^a, \tag{95}$$

$$\left\{\mathcal{H}_i, \underset{\sim}{\mathcal{H}_0}\right\} = \mathcal{H}_0\partial_i\delta^3(x-\widetilde{x}) + (...)^a\Phi^a, \tag{96}$$

$$\left\{\mathcal{H}_0, \underset{\sim}{\mathcal{H}_0}\right\} = \beta^{ik}\mathcal{H}_k\partial_i\delta^3(x-\widetilde{x}) - \underset{\sim}{\beta^{ik}}\underset{\sim}{\mathcal{H}_i}\partial_k\delta^3(x-\widetilde{x}) + (...)^a\Phi^a, \tag{97}$$

$$\left\{\Phi^a, \underset{\sim}{\mathcal{H}_i}\right\} = 0, \tag{98}$$

$$\left\{\Phi^a, \underset{\sim}{\mathcal{H}_0}\right\} = 0, \tag{99}$$

$$\left\{\Phi^a, \underset{\sim}{\Phi^b}\right\} = \varepsilon^{abc}\Phi^c\delta^3(x-\widetilde{x}). \tag{100}$$

Here all variables are to be expressed through the Q_a^i and $\mathcal{P}_l^b$.

It is essential that the transition from the Arnowitt-Deser-Misner (ADM) variables to FP ones does not change the algebra of constraints, because it is a canonical transformation. If we go to the tetrad formalism, the number of pairs of canonical variables rises, and new constraints appear. By (95)-(100) one sees that in the classical tetrad formalism all constraints are of the 1st class. The 1st-order Lagrangian for the closed universe must now be written in the form:

$$\mathcal{L}_{(1)}^{(\text{rep})} = \mathcal{P}_i^a\partial_0 Q_a^i - N\mathcal{H}_0 - N^i\mathcal{H}_i - \lambda^a\Phi^a, \tag{101}$$

where $\lambda^a \equiv \lambda^a(x) \equiv \lambda^a(x^0,x^1,x^2,x^3)|_{x^0=const}$ are new Lagrange multipliers, and the $\mathcal{H}_0$, $\mathcal{H}_i$ are the same quantities as in ADM or FP formalisms, but expressed through the Q_a^i and $\mathcal{P}_i^a$ by (66), (76).

The explicit form of the operators $\mathcal{H}_i$ and $\mathcal{H}_0$ is obtained in the FP formalism (see p. 7 of notes "Quantization of gravitation I. Metric tensor approach"). By (MT365), (MT339), (66), (76), (60), (62), (63) we find that

$$\begin{aligned}\mathcal{H}_i = -2q^{-\frac{1}{4}}q^{kl}\left(\overset{3}{\nabla}_i(q^{-\frac{1}{4}}\pi_{kl}) - \overset{3}{\nabla}_k(q^{\frac{1}{4}}\pi_{il})\right) = \\ = -2\beta^{-\frac{1}{2}}Q_b^kQ_b^l\left(\overset{3}{\nabla}_i(\beta^{\frac{1}{2}}\frac{1}{2}Q_l^a\mathcal{P}_k^a) - \overset{3}{\nabla}_k(\beta^{\frac{1}{2}}\frac{1}{2}Q_l^a\mathcal{P}_i^a)\right) = \\ = -Q^{\frac{1}{2}}Q_b^kQ_b^l\left(\overset{3}{\nabla}_i(Q^{\frac{1}{2}}Q_l^a\mathcal{P}_k^a) - \overset{3}{\nabla}_k(Q^{\frac{1}{2}}Q_l^a\mathcal{P}_i^a)\right) = -Q_b^ke_b^l\left(\overset{3}{\nabla}_i(e_l^a\mathcal{P}_k^a) - \overset{3}{\nabla}_k(e_c^a\mathcal{P}_i^a)\right) = \\ = -Q_b^ke_b^le_l^a\left(\overset{3}{\nabla}_i\mathcal{P}_k^a - \overset{3}{\nabla}_k\mathcal{P}_i^a\right) = Q_a^k\left(\overset{3}{\nabla}_k\mathcal{P}_i^a - \overset{3}{\nabla}_i\mathcal{P}_k^a\right);\end{aligned} \tag{102}$$

by (MT368), (MT339), (60), (63), (62), (66), (76)

$$\mathcal{H}_0 = \left(\frac{2\varkappa}{q^{\frac{1}{4}}}\right)(q^{lp}q^{mq} - q^{lm}q^{pq})\pi_{lm}\pi_{pq} - \left(\frac{q^{\frac{1}{4}}}{2\varkappa}\right)(\overset{3}{R} - 2\Lambda) =$$
$$= \left(\frac{2\varkappa}{\sqrt{\beta}}\right)(Q^l_a Q^p_a Q^m_b Q^q_b - Q^l_a Q^m_a Q^p_b Q^q_b)\left(\frac{1}{2}Q^c_l \mathcal{P}^c_m\right)\left(\frac{1}{2}Q^d_p \mathcal{P}^d_q\right) - \left(\frac{\sqrt{\beta}}{2\varkappa}\right)(\overset{3}{R} - 2\Lambda) =$$
$$= \frac{1}{4}\left(\frac{2\varkappa}{Q^{\frac{1}{2}}}\right)(Q^k_b Q^l_b \mathcal{P}^c_k \mathcal{P}^c_l - (Q^k_b \mathcal{P}^b_k)^2) - \left(\frac{q^{\frac{1}{2}}}{2\varkappa}\right)(\overset{3}{R} - 2\Lambda), \quad (103)$$

where the $\overset{3}{R}$ must be expressed in terms of the Q^k_a.

Let us represent the $\mathcal{H}_i$ in slightly different form. By (102)

$$\mathcal{H}_i = Q^{bk}(\overset{3}{\nabla}_k \mathcal{P}^b_i - \overset{3}{\nabla}_i \mathcal{P}^b_k) = Q^{bk}(\partial_k \mathcal{P}^b_i - \overset{3}{\Gamma}{}^l_{ki}\mathcal{P}^b_l + A^{bc}_k \mathcal{P}^c_i - \partial_i \mathcal{P}^b_k + \overset{3}{\Gamma}{}^l_{ik}\mathcal{P}^b_l - A^{bc}_k \mathcal{P}^c_k) =$$
$$= Q^{bk}(\partial_k \mathcal{P}^b_i - \partial_i \mathcal{P}^b_k) + (Q^{bk} A^{bc}_k)\mathcal{P}^c_i - A^c_i \varepsilon^{cab} Q^{ak}\mathcal{P}^b_k, \quad (104)$$

where we take

$$A^{ab}_i = A^c_i \varepsilon^{cab} \quad (105)$$

with the $A^{ab}_i = A_i{}^a{}_b$ being constructed from the e^a_i, e^i_k, δ^{ik}, δ_{ik} like the $A_\mu{}^A{}_B$ are constructed from the e^A_μ, e^μ_A, η^{AB}, η_{BA}. As it is seen from (25), (26), (37), (45), (48),

$$A^{ab}_i = A^{AB}_\mu\big|_{\mu=i,A=a,B=b}. \quad (106)$$

By (80)

$$\mathcal{H}_i = Q^{bk}(\partial_k \mathcal{P}^b_i - \partial_i \mathcal{P}^b_k) + (Q^{bk} A^{bc}_k)\mathcal{P}^c_i - A^c_i \Phi^c. \quad (107)$$

Further, by (106), (25), (26)

$$Q^{bk} A^{bc}_k = ee^{bk}(S^b_{lk} e^l_c - S^c_{lk} e^l_b - e^i_b S^d_{il} e^l_c e^d_k), \quad (108)$$

where it is taken into account (37), (102);

$$Q^{bk} A^{bc}_k = ee^{bk} S^b_{lk} e^l_c - ee^{bk} S^c_{lk} e^{bl} - ee^i_b S^b_{ie} e^l_c = 2ee^{bk} S^b_{lk} e^{lc} =$$
$$= ee^{bk}(\partial_l e^b_k - \partial_k e^b_l)e^{lc} = ee^{bk}(\partial_l e^b_k)e^{lc} - ee^{bk}(\partial_k e^b_l)e^{lc} =$$
$$= (\partial_l e)e^{lc} + ee^{bk} e^b_l \partial_k e^{lc} = (\partial_l e)e^{lc} + e\partial_l e^{lc} = \partial_l(ee^{lc}) = \partial_l Q^{lc}; \quad (109)$$

by (109), (107)

$$\mathcal{H}_i = Q^{bk}(\partial_k \mathcal{P}^b_i - \partial_i \mathcal{P}^b_k) + (\partial_l Q^{lb})\mathcal{P}^b_i - A^b_i \Phi^b, \quad (110)$$

i. e.

$$\mathcal{H}_i + A^c_i \Phi^c = Q^{bk}(\partial_k \mathcal{P}^b_i - \partial_i \mathcal{P}^b_k) + (\partial_k Q^{kb})\mathcal{P}^b_i. \quad (111)$$

The linear combination $\mathcal{H}_i + A^c_i \Phi^c$ of the constraints is also the constraint. It has the simple geometrical sense. Let us show that the $\mathcal{H}_i + A^c_i \Phi^c$ is the generator of such a 3-dimensional transformations of coordinates on the surface $x^0 = const$, that do not change tetrad basic vectors as geometrical objects. We have at infinitesimal $\varepsilon^i \equiv \varepsilon^i(x)$

$$\left\{\int d^3x\,\varepsilon^i(\mathcal{H}_i + A_i^c\Phi^c), \underset{\sim}{Q}^{ak}\right\} = \left\{\int d^3x(\partial_l(Q^{bl}\mathcal{P}_i^b) - Q^{bl}\partial_i\mathcal{P}_l^b)\varepsilon^i, \underset{\sim}{Q}^{ak}\right\} =$$
$$= -\int d^3x(\partial_l(Q^{bl}\delta^{ab}\delta_i^k\delta^3(x-\widetilde{x})) - Q^{bl}\partial_i(\delta^{ba}\delta_l^k\delta^3(x-\widetilde{x})))\varepsilon^i =$$
$$= -\int d^3x(\partial_l(Q^{al}\delta^3(x-\widetilde{x}))\varepsilon^k - Q^{ak}(\partial_i\delta^3(x-\widetilde{x}))\varepsilon^i) =$$
$$= \int d^3x(Q^{al}\partial_l\varepsilon^k - \partial_i(Q^{ak}\varepsilon^i))\delta^3(x-\widetilde{x}) = \underset{\sim}{Q}^{al}\underset{\sim}{\partial}_l\underset{\sim}{\varepsilon}^k - \underset{\sim}{\partial}_i(\underset{\sim}{Q}^{ak}\underset{\sim}{\varepsilon}^i); \quad (112)$$

$$\left\{\int d^3x\,\varepsilon^i(\mathcal{H}_i + A_i^c\Phi^c), \underset{\sim}{\mathcal{P}}_k^a\right\} = \left\{\int d^3x\,\varepsilon^i\left(\partial_l(Q^{bl}\mathcal{P}_i^b) - Q^{bl}\partial_i\mathcal{P}_l^b\right), \underset{\sim}{\mathcal{P}}_k^a\right\} =$$
$$= \int d^3x\,\varepsilon^i\left(\partial_l(\delta^{ba}\delta_k^l\delta^3(x-\widetilde{x})\mathcal{P}_i^b) - \delta^{ba}\delta_k^l\delta^3(x-\widetilde{x})\partial_i\mathcal{P}_l^b\right) =$$
$$= \int d^3x\,\left(\varepsilon^i\partial_k(\mathcal{P}_i^a\delta^3(x-\widetilde{x})) - \varepsilon^i(\partial_i\mathcal{P}_k^a)\delta^3(x-\widetilde{x})\right) =$$
$$= -\int d^3x\,(\mathcal{P}_i^a(\partial_k\varepsilon^i) + (\partial_i\mathcal{P}_k^a)\varepsilon^i)\delta^3(x-\widetilde{x}) = -\underset{\sim}{\mathcal{P}}_i^a\underset{\sim}{\partial}_k\underset{\sim}{\varepsilon}^i - (\underset{\sim}{\partial}_i\underset{\sim}{\mathcal{P}}_k^a)\underset{\sim}{\varepsilon}^i. \quad (113)$$

Thus,

$$\left\{\int d^3x\,\varepsilon^i(\mathcal{H}_i + A_i^b\Phi^b), \underset{\sim}{Q}^{ak}\right\} = \underset{\sim}{Q}^{al}\underset{\sim}{\partial}_l\underset{\sim}{\varepsilon}^k - \underset{\sim}{Q}^{ak}\underset{\sim}{\partial}_i\underset{\sim}{\varepsilon}^i - \underset{\sim}{\varepsilon}^i\underset{\sim}{\partial}_i\underset{\sim}{Q}^{ak}, \quad (114)$$

$$\left\{\int d^3x\,\varepsilon^i(\mathcal{H}_i + A_i^b\Phi^b), \underset{\sim}{\mathcal{P}}_k^a\right\} = -\underset{\sim}{\mathcal{P}}_i^a\underset{\sim}{\partial}_k\underset{\sim}{\varepsilon}^i - \underset{\sim}{\varepsilon}^i\underset{\sim}{\partial}_i\underset{\sim}{\mathcal{P}}_k^a. \quad (115)$$

On the other side, by the transformation law of 3-tensors, as $x^i \to x'^i = x^i + \varepsilon^i(x)$, while unmoved vectors of tetrad basis, taking into account (MT232) one gets

$$Q'^{ak}(x) = Q'^{ak}(x'-\varepsilon) = -\varepsilon^i\partial_i Q^{ak} + Q'^{ak}(x') =$$
$$= -\varepsilon^i\partial_i Q^{ak} + \sqrt{\beta'(x')}(e'^{ak}(x')) = -\varepsilon^i\partial_i Q^{ak} + \sqrt{\beta(x)}(1-\partial_i\varepsilon^i)\left(e^{al}(x)\frac{\partial(x^k+\varepsilon^k)}{\partial x^l}\right) =$$
$$= -\varepsilon^i\partial_i Q^{ak} + \sqrt{\beta(x)}e^{al}(x) - \sqrt{\beta}e^{ak}\partial_i\varepsilon^i - \sqrt{\beta}e^{al}\partial_l\varepsilon^k =$$
$$= Q^{ak}(x) + Q^{al}\partial_l\varepsilon^k - Q^{ak}\partial_i\varepsilon^i - \varepsilon^i\partial_i Q^{ak}. \quad (116)$$

Since, further, the $\mathcal{P}_i^a Q_a^i$ transform as $\sqrt{\beta}$, and $Q_a^i = \sqrt{\beta}e_a^i$, the $\mathcal{P}_i^a$ is a universal 3-tensor. Therefore, at the considered change of coordinates,

$$\mathcal{P}'^a_k(x) = -\varepsilon^i\partial_i\mathcal{P}_k^a + \mathcal{P}'^a_k(x') = -\varepsilon^i\partial_i\mathcal{P}_k^a + \mathcal{P}_l^a(x)\frac{\partial(x'^l-\varepsilon^l)}{\partial x'^k} =$$
$$= -\mathcal{P}_i^a(x)\partial_k\varepsilon^i - \varepsilon^i\partial_i\mathcal{P}_k^a + \mathcal{P}_k^a(x). \quad (117)$$

So, at the transformation of coordinates

$$\delta Q^{ak} \equiv Q'^{ak}(x) - Q^{ak}(x) = Q^{al}\partial_l\varepsilon^k - Q^{ak}\partial_i\varepsilon^i - \varepsilon^i\partial_i Q^{ak}, \quad (118)$$

$$\delta \mathcal{P}^a_k \equiv \mathcal{P}'^a_k(x) - \mathcal{P}^a_k(x) = -\mathcal{P}^a_i(x)\partial_k \varepsilon^i - \varepsilon^i \partial_i \mathcal{P}^a_k. \tag{119}$$

Comparing (114), (115) with the (118), (119), we see that the quantities $\mathcal{H}_i + A^b_i\Phi^b$ are indeed the generators of the such 3-dimensional transformations of coordinates on the surface $x^0 = const$, which do not change the position of the vierbein basic vectors in the space. W. r. t. these transformations the $\mathcal{H}_0$ is a stable density of a 3-invariant, and the $\mathcal{H}_i$ and $A^b_i\Phi^b$ are the stable densities of coordinate 3-vectors. So, repeating the derivation of the equalities (MT268), (MT269), we find that

$$\left\{\mathcal{H}_i + A^b_i\Phi^b, \underset{\sim}{\mathcal{H}}_k\right\} = -\underset{\sim}{\mathcal{H}}_i \underset{\sim}{\partial}_k \delta^3(x - \widetilde{x}) + \mathcal{H}_k\partial_i\delta^3(x - \widetilde{x}), \tag{120}$$

$$\left\{\mathcal{H}_i + A^b_i\Phi^b, \underset{\sim}{A}{}^c_k \underset{\sim}{\Phi}{}^c\right\} = -\underset{\sim}{A}{}^c_i \underset{\sim}{\Phi}{}^c \underset{\sim}{\partial}_k \delta^3(x - \widetilde{x}) + A^c_k\Phi^c\partial_i\delta^3(x - \widetilde{x}), \tag{121}$$

$$\left\{\mathcal{H}_i + A^b_i\Phi^b, \underset{\sim}{\mathcal{H}}_0\right\} = \mathcal{H}_0\partial_i\delta^3(x - \widetilde{x}). \tag{122}$$

It follows from (120), (122), due to (98), (99), that

$$\left\{\mathcal{H}_i, \underset{\sim}{\mathcal{H}}_k\right\} = -\underset{\sim}{\mathcal{H}}_i \underset{\sim}{\partial}_k \delta^3(x - \widetilde{x}) + \mathcal{H}_k\partial_i\delta^3(x - \widetilde{x}) - \left\{A^b_i, \underset{\sim}{\mathcal{H}}_k\right\}\Phi^b, \tag{123}$$

$$\left\{\mathcal{H}_i, \underset{\sim}{\mathcal{H}}_0\right\} = \mathcal{H}_0\partial_i\delta^3(x - \widetilde{x}) - \left\{A^b_i, \underset{\sim}{\mathcal{H}}_0\right\}\Phi^b. \tag{124}$$

This is a detailed elaboration of the formulas (95), (96). It follows also from (86), (89) that

$$\left\{\mathcal{H}_i + A^b_i\Phi^b, \underset{\sim}{\mathcal{H}}_k + \underset{\sim}{A}{}^c_k\underset{\sim}{\Phi}{}^c\right\} = -(\underset{\sim}{\mathcal{H}}_i + \underset{\sim}{A}_i\underset{\sim}{\Phi}{}^c)\underset{\sim}{\partial}_k\delta^3(x - \widetilde{x}) + (\mathcal{H}_k + A^c_k\Phi^c)\partial_i\delta^3(x - \widetilde{x}). \tag{125}$$

By (81) it can be written, further,

$$\left\{\int d^3x A^b_i\Phi^b\varepsilon^i, \underset{\sim}{Q}{}^{ld}\right\} = \int d^3x A^b_i\varepsilon^i\left\{\Phi^b, \underset{\sim}{Q}{}^{ld}\right\} = \varepsilon^{dba}\underset{\sim}{Q}{}^{ld}\underset{\sim}{A}{}^a_i\underset{\sim}{\varepsilon}_i = \underset{\sim}{\varepsilon}{}^i\underset{\sim}{A}{}^{db}_i\underset{\sim}{Q}{}^{lb}, \tag{126}$$

analogously,

$$\left\{\int d^3x\, \varepsilon^i A^b_i\Phi^b, \underset{\sim}{\mathcal{P}}{}^a_k\right\} = \int d^3x\, \varepsilon^i\left\{A^b_i, \underset{\sim}{\mathcal{P}}{}^a_k\right\}\Phi^b + \underset{\sim}{\varepsilon}{}^i\underset{\sim}{A}{}^{ab}_i\underset{\sim}{\mathcal{P}}{}^b_k. \tag{127}$$

By analogy with the $\{\mathcal{H}_i + A^b_i\Phi^b, \underset{\sim}{\mathcal{H}}_0\}$ we have

$$\left\{\mathcal{H}_i + A^b_i\Phi^b, \underset{\sim}{\Phi}{}^a\right\} = \Phi^a\partial_i\delta^3(x - \widetilde{x}). \tag{128}$$

From the equalities (114), (115) and (126), (127) we find that

$$\left\{\int d^3x\, \varepsilon^i\mathcal{H}_i, \underset{\sim}{Q}{}^{ak}\right\} = \underset{\sim}{Q}{}^{al}\underset{\sim}{\partial}_l\underset{\sim}{\varepsilon}{}^k - \underset{\sim}{Q}{}^{ak}\underset{\sim}{\partial}_i\underset{\sim}{\varepsilon}{}^i - \underset{\sim}{\varepsilon}{}^i(\underset{\sim}{\partial}_i\underset{\sim}{Q}{}^{ak} + \underset{\sim}{A}{}^{ab}_i\underset{\sim}{Q}{}^{kb}), \tag{129}$$

$$\left\{\int d^3x\, \varepsilon^i\mathcal{H}_i, \underset{\sim}{\mathcal{P}}{}^a_k\right\} = -\underset{\sim}{\mathcal{P}}{}^a_l\underset{\sim}{\partial}_k\underset{\sim}{\varepsilon}{}^l - \underset{\sim}{\varepsilon}{}^i(\underset{\sim}{\partial}_i\underset{\sim}{\mathcal{P}}{}^a_k + \underset{\sim}{A}{}^{ab}_i\underset{\sim}{\mathcal{P}}{}^b_k) - \int d^3x\left(\varepsilon^i\left\{A^b_i, \underset{\sim}{\mathcal{P}}{}^a_k\right\}\Phi^b\right). \tag{130}$$

The formulae (129) differs from (114) by the change of the usual derivative $\partial_i Q^{ak}$ onto covariant one only in tetrad index. This means that the quantities $\mathcal{H}_i$ are the generators of the such variations of the functions Q^{ib}, which are generated by the change of the coordinates, carrying with itself the vierbein system via parallel transfer. The same follows from (130) w. r. t. the $\mathcal{P}^a_k$, if one adopts the constraint $\Phi^a = 0$.

3 Tetrad variables having the property of connection, "loop variables"

With the help of canonical transformation one can go from the variables Q^i_a, $\mathcal{P}^a_i$, introduced above, to the variables, having the form of a connection. This allows then to apply the methods, used in the gauge field theory, what leads to the so called "loop quantum gravity theory". Let us consider the main ideas related to this formalism. We use the vierbein frame, which is related to the coordinate frame as described in sec. 2. Let us start from dynamical variables Q^i_a, $\mathcal{P}^a_i$, introduced in (57) and (75). First of all, let us construct the appropriate classical canonical formalism.

Earlier (see eq-ns (25), (26)), the following formula for the coefficients of vierbein connection was obtained:

$$A_{\mu,AB} \equiv \eta_{AD} A^{D}_{\mu\ B} = -A_{\mu,BA} = \eta_{AD} S^{D}_{\alpha\mu} e^{\alpha}_{B} - \eta_{BD} S^{D}_{\alpha\mu} e^{\alpha}_{A} - e^{E}_{\mu} \eta_{ED} e^{\alpha}_{A} S^{D}_{\alpha\beta} e^{\beta}_{B}, \tag{131}$$

where

$$S^{D}_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha e^{D}_{\beta} - \partial_\beta e^{D}_{\alpha}\right). \tag{132}$$

In the used vierbein frame, where $e^{\mu}_{(0)} = n^{\mu}$, $\quad e^{0}_{a} = 0$, $\quad e^{(0)}_{i} = 0$, the 3-dimensional part of the connection (131) has the form

$$A_{i,ab} = \eta_{aD} S^{D}_{\alpha i} e^{\alpha}_{b} - \eta_{bD} S^{D}_{\alpha i} e^{\alpha}_{a} - e^{E}_{i} \eta_{ED} e^{\alpha}_{a} S^{D}_{\alpha\beta} e^{\beta}_{b} = \delta_{ad} S^{d}_{ki} e^{k}_{b} - \delta_{bd} S^{d}_{ki} e^{k}_{a} - e^{c}_{i} \delta_{cd} e^{k}_{a} S^{d}_{kl} e^{l}_{b}, \tag{133}$$

i. e.

$$A_{i,ab} = S^{a}_{ki} e^{k}_{b} - S^{b}_{ki} e^{k}_{a} - e^{c}_{i} e^{k}_{a} S^{c}_{kl} e^{l}_{b}. \tag{134}$$

Here

$$S^{a}_{ki} = \frac{1}{2}\left(\partial_k e^{a}_{i} - \partial_i e^{a}_{k}\right). \tag{135}$$

It is taken into account that 3-dimensional indices $a, b, \ldots$ are lifted up or pulled down with the help of the symbols δ^{ab}, δ_{ab} , and therefore there is no difference between up and down indices $a, b, \ldots$. It is seen that with our choice of vierbein frame the 3-dimensional part $A_{i,ab}$ of the connection is constructed from vierbein parameters e^{i}_{a},e^{a}_{i} and symbols δ_{ab}, exactly in the same way as the connection $A_{m,AD}$ is composed from e^{m}_{A}, e^{B}_{m} and the symbols η_{AB}, i. e.

$$A_{i,ab} = \overset{3}{A}_{i,ab}, \tag{136}$$

where the $\overset{3}{A}_{i,ab}$ is vierbein connection on the 3-dimensional hypersurface $x^0 = const$, corresponding to the local invariance group $SO(3)$. Accordingly,

$$A_{i,ab} = A_{i\ b}^{\ a} = A_{i}^{\ ab} = -A_{i,ba}. \tag{137}$$

Obviously, one can also write

$$A_{i}^{\ ab} = \varepsilon^{abc} A^{c}_{i}, \tag{138}$$

where

$$A_i^c = \frac{1}{2}\varepsilon^{cab} A_i{}^{ab} = \frac{1}{2}\varepsilon^{cab}\left(2S^a_{ki}e^k_b - e^c_i e^k_a S^c_{kl} e^l_b\right). \tag{139}$$

Here we have taken into account that $\varepsilon^{cab}S^a_{ki} = -\varepsilon^{cab}S^b_{ki}e^k_a$ and used the (133).

Let us express the quantities A^c_i in terms of the variables Q^i_a, Q^a_i. By the definitions, introduced above

$$e^i_a = Q^{-\frac{1}{2}}Q^i_a, \quad e^a_i = Q^{\frac{1}{2}}Q^a_i, \tag{140}$$

where

$$Q = \det\left(Q^i_a\right), \qquad Q^i_a Q^a_i = \delta^i_k. \tag{141}$$

Evidently,

$$\partial_i Q^{\frac{1}{2}} = \frac{1}{2}Q^{-\frac{1}{2}}QQ^a_k\partial_i Q^k_a = \frac{1}{2}Q^{\frac{1}{2}}Q^a_k\partial_i Q^k_a, \tag{142}$$

$$\partial_i Q^a_k = -Q^a_l\left(\partial_i Q^l_b\right)Q^b_k. \tag{143}$$

The equality (143) is obtained by the differentiation of the relation $Q^a_k Q^k_b = \delta^a_b$, and the (142) is true because

$$\det(Q^i_a + dQ^i_a) = \left(\text{the algebraic complement of the matrix } Q^l_b\right)^a_i dQ^i_a = QQ^a_i dQ^i_a. \tag{144}$$

By (139), (135), (140), (142) we obtain

$$2A^c_i = \varepsilon^{cab}\left(\left(\partial_k e^a_i - \partial_i e^a_k\right)e^k_b - e^k_a\frac{1}{2}\left(\partial_k e^d_l - \partial_l e^d_k\right)e^l_b e^d_i\right) =$$
$$= \frac{1}{4}\varepsilon^{cab}\Big(4Q^k_b\partial_k Q^a_i + 2Q^a_i Q^k_b Q^f_m\partial_k Q^m_f - 4Q^k_b\partial_i Q^a_k - 2\delta^a_b Q^f_m\partial_i Q^m_f -$$
$$-Q^b_i Q^k_a Q^f_m\partial_k Q^m_f - 2Q^k_a(\partial_k Q^d_l)Q^l_b Q^d_i + Q^a_i Q^l_b Q^f_m\partial_l Q^m_f + 2Q^k_a(\partial_l Q^d_k)Q^l_b Q^d_i\Big). \tag{145}$$

Due to $\varepsilon^{cab} = -\varepsilon^{cba}$, the term, containing the δ^a_b, does not contribute here, and some other terms coincide. Therefore,

$$A^c_i = \frac{1}{2}\varepsilon^{cab}\left(Q^a_i Q^k_b Q^d_l\partial_k Q^l_d + Q^l_a Q^k_b Q^d_i\partial_k Q^d_l + Q^k_b\partial_k Q^a_i + Q^a_k\partial_i Q^k_b\right). \tag{146}$$

Studying the properties of the continuation of the gravitational field into the complex region (that we will concern later), A. Ashtekar has found that the following change of variables is a canonical one:

$$Q'^i_a = Q^i_a, \tag{147}$$

$$\mathcal{P}'^a_i = \mathcal{P}^a_i + bA^a_i, \tag{148}$$

where the b is an arbitrary constant parameter, i. e. that

$$\left\{Q'^i_a, \underset{\sim}{Q}'^k_b\right\} = \left\{Q^i_a, \underset{\sim}{Q}^k_a\right\} = 0, \tag{149}$$

$$\left\{Q'^{i}_{a}, \underset{\sim}{\mathcal{P}}'^{b}_{a}\right\} = \left\{Q^{i}_{a}, \underset{\sim}{\mathcal{P}}^{b}_{k}\right\} = \delta^{i}_{k}\delta^{b}_{a}\delta^{3}(x-\tilde{x}), \tag{150}$$

$$\left\{\mathcal{P}'^{a}_{i}, \underset{\sim}{\mathcal{P}}'^{b}_{k}\right\} = \left\{\mathcal{P}^{a}_{i}, \underset{\sim}{\mathcal{P}}^{b}_{k}\right\} = 0. \tag{151}$$

Since the A^a_i depends only on the Q^l_a,and not on the $\mathcal{P}^a_l$, the equalities (149) and (150) are fulfilled trivially, but the correctness of the relation (151) is a very nontrivial fact. Indeed,

$$\left\{\mathcal{P}'^{a}_{i}, \underset{\sim}{\mathcal{P}}'^{b}_{k}\right\} = \left\{\mathcal{P}^{a}_{i} + bA^{a}_{i}, \underset{\sim}{\mathcal{P}}^{b}_{k} + b\underset{\sim}{A}^{b}_{k}\right\} = \left\{\mathcal{P}^{a}_{i}, \underset{\sim}{\mathcal{P}}^{b}_{k}\right\} + b\left\{A^{a}_{i}, \underset{\sim}{\mathcal{P}}^{b}_{k}\right\} + b\left\{\mathcal{P}^{a}_{i}, \underset{\sim}{A}^{b}_{k}\right\}. \tag{152}$$

We have taken into account that $\left\{A^{a}_{i}, \underset{\sim}{A}^{b}_{k}\right\} = 0$ because the A^a_i depends only on the Q^i_a.

Thus, in order that the transformation (147), (148) were canonical, the following equality must be fulfilled:

$$\left\{\mathcal{P}^{a}_{i}, \underset{\sim}{A}^{b}_{k}\right\} + \left\{A^{a}_{i}, \underset{\sim}{\mathcal{P}}^{b}_{k}\right\} \equiv \left\{\mathcal{P}^{a}_{i}, \underset{\sim}{A}^{b}_{k}\right\} - \left\{\underset{\sim}{\mathcal{P}}^{b}_{k}, A^{a}_{i}\right\} = 0. \tag{153}$$

By the definition of Poisson brackets one has

$$\left\{\mathcal{P}^{a}_{i}, \underset{\sim}{A}^{b}_{k}\right\} = \int\limits_{t'=const} d^{3}x' \left(\frac{\delta\mathcal{P}^{a}_{i}(x)}{\delta Q^{l}_{c}(x')}\frac{\delta A^{b}_{k}(\widetilde{x})}{\delta\mathcal{P}^{l}_{c}(x')} - \frac{\delta A^{b}_{k}(\widetilde{x})}{\delta Q^{l}_{c}(x')}\frac{\delta\mathcal{P}^{a}_{i}(x)}{\delta\mathcal{P}^{c}_{l}(x')}\right), \tag{154}$$

where $x \equiv (x^1, x^2, x^3)$, and the $\frac{\delta(\ldots)}{\delta(\ldots)}$ means the 3-dimensional functional derivative. Owing to

$$\frac{\delta\mathcal{P}^{a}_{i}(x)}{\delta Q^{l}_{c}(x')} = 0, \qquad \frac{\delta\mathcal{P}^{a}_{i}(x)}{\delta\mathcal{P}^{c}_{l}(x')} = \delta^{a}_{c}\delta^{l}_{i}\delta(x-x'), \tag{155}$$

we get

$$\left\{\mathcal{P}^{a}_{i}, \underset{\sim}{A}^{l}_{k}\right\} = -\frac{\delta A^{b}_{k}(\widetilde{x})}{\delta Q^{l}_{a}(x)}. \tag{156}$$

Thus, the relation (153) takes the form

$$\frac{\delta A^{b}_{k}(\widetilde{x})}{\delta Q^{a}_{i}(x)} - \frac{\delta A^{a}_{i}(x)}{\delta Q^{b}_{k}(\widetilde{x})} = 0. \tag{157}$$

As in the case of usual functions $f_k(x)$, when the equality $\partial_i f_k - \partial_k f_i = 0$ is equivalent to the existence of such a $\varphi(x)$ that $f_i = \partial_i\varphi$, the relation (157) is fulfilled then and only then when there exists a functional $F[Q^i_a]$ of functions $Q^i_a(x)$ for which

$$A^{a}_{i}(x) = \frac{\delta F}{\delta Q^{i}_{a}(x)}. \tag{158}$$

Such a functional does exist and is equal to

$$F = \frac{1}{2}\varepsilon^{cab} \int\limits_{t=const} d^{3}x\, Q^{i}_{c}Q^{a}_{k}\partial_{i}Q^{k}_{b}. \tag{159}$$

The possibility to represent the complicated expression like (146) in the form (158), (159) is highly nontrivial. A. Ashtekar discovered this, going by circuitous way (this will be considered briefly later). Now we simply check the equality (158) at the condition (159) directly. Let us make this.

Under the variation δQ^i_c of the field Q^i_c the variation of the functional $F[Q^i_c(x)]$ is determined, in accordance with (159), by the equality

$$\begin{aligned} 2\delta F = \varepsilon^{cab}\int d^3x\,\left(\delta Q^i_c Q^a_k \partial_i Q^k_b + Q^i_c \delta Q^a_k \partial_i Q^k_b + Q^i_c Q^a_k \partial_i \delta Q^k_b\right) = \\ = \int d^3x\,\left(\varepsilon^{cab}\delta Q^i_c Q^a_k \partial_i Q^k_b + \varepsilon^{dab} Q^l_d(-Q^a_i \delta Q^i_c Q^c_k)\partial_l Q^k_b + \varepsilon^{bac} Q^k_b Q^a_i \partial_k \delta Q^i_c\right) = \\ = \int d^3x\,\left(\varepsilon^{cab} Q^a_k \partial_i Q^k_b - \varepsilon^{dfg} Q^l_d Q^f_i Q^c_k \partial_l Q^k_g + \varepsilon^{cab}(\partial_k Q^k_b Q^a_i + Q^k_b \partial_k Q^a_i)\right)\delta Q^i_c. \quad (160) \end{aligned}$$

Here we use (143) and perform the change

$$\varepsilon^{bac} Q^k_b Q^a_i \partial_k \delta Q^i_c \longrightarrow -\varepsilon^{bac}\left(\partial_k (Q^k_b Q^a_i)\right)\delta Q^i_c = \varepsilon^{cab}\left((\partial_k Q^k_b) Q^a_c + Q^k_b \partial_k Q^a_i\right)\delta Q^i_c, \quad (161)$$

dropping the nonessential here surface term in the integral. Therefore

$$2\frac{\delta F}{\delta Q^i_c} = \varepsilon^{cab}\left(Q^a_k \partial_i Q^k_b + Q^a_i \partial_k Q^k_b + Q^k_b \partial_k Q^a_i\right) - \varepsilon^{dfg} Q^l_d Q^f_i Q^c_k \partial_l Q^k_g. \quad (162)$$

Let us show that this coincides with the (146). Take into account that $\frac{1}{2}\varepsilon^{cab}\varepsilon^{abh} = \delta^{ch}$, $\varepsilon^{cab} = -\varepsilon^{cba}$ and, hence,

$$\begin{aligned} -\varepsilon^{dfg} Q^c_k = -\frac{1}{2}\varepsilon^{cab}\varepsilon^{dfg}\varepsilon^{abh} Q^h_k = \frac{1}{2}\varepsilon^{cab}\varepsilon^{dfg}\varepsilon^{bah} Q^h_k = \\ = \frac{1}{2}\varepsilon^{cab}\left(\delta^{db}\delta^{fa}\delta^{gh} + \delta^{da}\delta^{fh}\delta^{gb} + \delta^{dh}\delta^{fb}\delta^{ga} - \delta^{da}\delta^{fb}\delta^{gh} - \delta^{db}\delta^{fh}\delta^{ga} - \delta^{dh}\delta^{fa}\delta^{gb}\right) Q^h_k = \\ = \varepsilon^{cab}\left(\delta^{db}\delta^{fa}\delta^{gh} + \delta^{da}\delta^{fh}\delta^{gb} + \delta^{dh}\delta^{fb}\delta^{ga}\right) Q^h_k. \quad (163) \end{aligned}$$

We used the known formula expressing the $\varepsilon^{dfg}\varepsilon^{bah}$ in terms of the products of the δ^{ab}-symbols. By (162), (163) we have

$$\begin{aligned} 2\frac{\delta F}{\delta Q^i_c} = \varepsilon^{cab}\big(Q^a_k \partial_i Q^k_b + Q^a_i \partial_k Q^k_b + Q^k_b \partial_k Q^a_i + \\ + \left(\delta^{db}\delta^{fa}\delta^{gh} + \delta^{da}\delta^{fh}\delta^{gb} + \delta^{dh}\delta^{fb}\delta^{ga}\right) Q^l_d Q^f_i Q^h_k \partial_l Q^k_g\big) = \\ = \varepsilon^{cab}\big(Q^a_k \partial_i Q^k_b + Q^a_i \partial_k Q^k_b + Q^k_b \partial_k Q^a_i + \\ + Q^l_b Q^a_i Q^h_k \partial_l Q^k_h + Q^l_a Q^h_i Q^h_k \partial_k Q^k_b + Q^l_h Q^b_l Q^h_k \partial_l Q^k_a\big). \quad (164) \end{aligned}$$

Evidently,

$$Q^l_h Q^b_i Q^h_k \partial_l Q^k_a = Q^b_i \delta^l_k \partial_l Q^k_a = Q^b_i \partial_k Q^k_a, \qquad \varepsilon^{cab}\left(Q^a_i \partial_k Q^k_b + Q^b_i \partial_k Q^k_a\right) = 0. \quad (165)$$

Therefore some terms in the (164) are cancelled. Further,

$$\varepsilon^{cab} Q^l_a Q^h_i Q^h_k \partial_l Q^k_b = -\varepsilon^{cab} Q^l_a Q^h_i \partial_l Q^h_k Q^k_b = \varepsilon^{cab} Q^l_b Q^h_i Q^k_a \partial_l Q^h_k. \quad (166)$$

Hence,

$$\frac{\delta F}{\delta Q^i_c} = \frac{1}{2}\varepsilon^{cab}\left(Q^a_k \partial_i Q^k_b + Q^k_b \partial_k Q^a_i + Q^l_b Q^a_i Q^h_k \partial_l Q^k_h + Q^l_b Q^h_i Q^k_a \partial_l Q^h_k\right). \quad (167)$$

That coincides with the (146). Thus the equality (158) is proven under the condition (159), and, therefore, the relations (147), (148) define the canonical transformation.

Having performed the canonical transformation (147), (148) we can then perform other two transformations, canonical character of which is evident. Let us put

$$Q''^i_a = bQ'^i_a = bQ^i_a, \tag{168}$$

$$\mathcal{P}''^a_i = \frac{1}{b}\mathcal{P}'^a_i = A^a_i + \frac{1}{b}\mathcal{P}^a_i, \tag{169}$$

and then

$$B^a_i = \mathcal{P}''^a_i = A^a_i + \frac{1}{b}\mathcal{P}^a_i, \tag{170}$$

$$\Pi^i_a = -Q''^i_a = -bQ^i_a. \tag{171}$$

We will consider the B^a_i as new canonical coordinates, while the Π^i_a as canonical momenta. The transition from the Q^i_a, $\mathcal{P}^a_i$ to the B^a_i, Π^i_a is a canonical transformation. This transformation was proposed by A. Ashtekar. The constant b is called the Barbero-Immirzi parameter. It can take any value.

Let us clear up how the quantity $\mathcal{P}^a_i$ transforms under the change of coordinates, and also under the change of vierbeins, which does not violate the condition $e^\mu_{(0)} = n^\mu$. By (76)

$$\mathcal{P}^a_i = 2\pi_{ik}Q^k_a, \tag{172}$$

and by the (MT357)

$$\pi_{ik} = -\frac{1}{(2\varkappa)2\sqrt{\beta}}\mathcal{J}_{ik,lm}P^{lm}. \tag{173}$$

and in accordance with (MT163)

$$P^{lm} = -2\mathcal{J}^{lm,rs}K_{rs}. \tag{174}$$

Due to

$$\mathcal{J}_{ik,lm}\mathcal{J}^{lm,rs} = \delta^{rs}_{lm} = \frac{1}{2}\left(\delta^r_l\delta^s_m + \delta^r_m\delta^s_l\right), \tag{175}$$

we get

$$\pi_{ik} = \left(-\frac{1}{(2\varkappa)2\sqrt{\beta}}\mathcal{J}_{ik,lm}\right)\left(-2\mathcal{J}^{lm,rs}\right)K_{rs} = \frac{1}{2\varkappa\sqrt{\beta}}K_{ik} \tag{176}$$

and in accord with (172)

$$\mathcal{P}^a_i = 2\left(\frac{1}{(2\varkappa)\sqrt{\beta}}K_{ik}\right)Q^k_a = \frac{2}{2\varkappa}\frac{Q^k_a}{\sqrt{\beta}}K_{ik} \tag{177}$$

or

$$\mathcal{P}^a_i = \frac{2}{2\varkappa}e^k_a K_{ki}. \tag{178}$$

Since the e^k_a, K_{ki} are stable 3-dimensional tensors w. r. t. coordinate transformations and tensors (in particular, vector and invariant) w. r. t. the 3-dimensional vierbein $SO(3)$ – transformations, the $\mathcal{P}^a_i$ is a stable coordinate vector and a vierbein $SO(3)$- vector. Therefore, the quantity

$$\mathcal{P}^{ab}_i \equiv \mathcal{P}^a_{i\,b} = \varepsilon^{abc}\mathcal{P}^c_i \tag{179}$$

is a vierbein $SO(3)$ – tensor and changes under the $SO(3)$ – transformations of vierbeins according to the rule

$$\widehat{\mathcal{P}}'_i = \widehat{\omega}\widehat{\mathcal{P}}_i\widehat{\omega}^{-1}, \tag{180}$$

where the $\widehat{\mathcal{P}}_i$ is the matrix with elements $\mathcal{P}_i{}^a{}_b$, and the $\widehat{\omega}$ is the matrix of $SO(3)$ – transformation, such that

$$\widehat{\omega}\widehat{\omega}^T = I. \tag{181}$$

At the same time the quantity

$$A^a_{i\,b} \equiv A_i{}^{ab} = \varepsilon^{abc}A^c_i \tag{182}$$

is a 3-dimensional $SO(3)$- connection (as was clarified above). So, it transforms under the $SO(3)$ – transformations of vierbeins as follows:

$$\widehat{A}'_i = \widehat{\omega}\widehat{A}_i\widehat{\omega}^{-1} + \widehat{\omega}\partial_i\widehat{\omega}^{-1}. \tag{183}$$

This formulae is completely analogous to the corresponding relation of the gauge field theory (with the replacing of $SU(3)$-matrices by $SO(3)$-matrices, and 4-dimensional space-time by 3-dimensional space). The formulae (183) can be got from the requirement that the covariant derivative of the vierbein vector

$$\overset{3}{\nabla}_i a^a = \partial_i a^a + A_i{}^a{}_b\, a^b \tag{184}$$

has to be a vierbein vector, i. e. that the following relation has to be valid:

$$\overset{3}{\nabla}_i(\omega^a{}_b a^b) = \omega^a{}_b \overset{3}{\nabla}_i a^b. \tag{185}$$

Beside of this, the coefficients of the connection $A_i{}^a{}_b$ are stable vectors w. r. t. 3-dimensional coordinate transformations.

Let us form now the matrix $\widehat{B}_i$ with the elements

$$B^a_{i\,b} = B_i{}^{ab} = \varepsilon^{abc}B^c_i. \tag{186}$$

By (170) one has

$$\widehat{B}_i = \widehat{A}_i + \frac{1}{b}\widehat{\mathcal{P}}_i \tag{187}$$

and, by the (187), (180), (183), under the $SO(3)$ – transformation of vierbeins, one gets

$$\widehat{B}'_i = \widehat{\omega}\widehat{B}_i\widehat{\omega}^{-1} + \widehat{\omega}\partial_i\widehat{\omega}^{-1}. \tag{188}$$

In other words, the quantities $B_i{}^a{}_b$ transform under the $SO(3)$ – change of vierbeins in the same way as the $A_i{}^a{}_b$, i. e. the $B_i{}^a{}_b$ are the coefficients of new $SO(3)$ – connection. This transition to the generalized coordinates, having the character of the connection, was the goal of the whole construction. Let us remark that under the 3-dimensional coordinate transformations the quantity $B_i{}^a{}_b$ behaves as a stable vector because the $A_i{}^a{}_b$ and the $\mathcal{P}_i{}^a{}_c$ have this property.

With the help of the connection $\widehat{B}_i$ one can construct contour integrals in the way, analogous to the construction of the so called Wilson-Polyakov integrals in gauge theory. Let us define, at first, the notion of contour integral with the connection $\widehat{A}_i$. Let us parameterize some curve Γ (contour) on the hypersurface $t = const$ with the help of the equality $x^i = f^i(\xi)$ where the ξ is the parameter varying along the contour. We assume that the derivatives $\partial x^i/\partial\xi$ do not be equal to zero simultaneously nowhere.

Let $a^a(\xi)$ to be a vierbein vector on the contour Γ at the point ξ. We define a parallel transportation of a vector $a^a(\xi)$ to infinitesimally close point $\xi + d\xi$ of the contour Γ so that the quantity $a^a(\xi + d\xi)$ resulting under the transportation be a vector at the point $\xi + d\xi$. This condition is fulfilled if one takes

$$a^a(\xi + d\xi) = a^a(\xi) - A^a_{i\,b}(x(\xi))a^b(\xi)\frac{dx^i(\xi)}{d\xi}d\xi. \tag{189}$$

Indeed, let us write the equality (189) in the form

$$a(\xi + d\xi) = a(\xi) - \widehat{A}_i(x(\xi))a(\xi)\frac{dx^i(\xi)}{d\xi}d\xi \tag{190}$$

and take into account that under the change of vierbeins we get

$$a'(\xi) = \widehat{\omega}(x(\xi))a(\xi), \tag{191}$$

$$\begin{aligned}\widehat{A}'_i(x(\xi)) &= \widehat{\omega}(x(\xi))\widehat{A}_i(x(\xi))\widehat{\omega}^{-1}(x(\xi)) + \widehat{\omega}(x(\xi))\frac{\partial\widehat{\omega}^{-1}(x)}{\partial x^i}\bigg|_{x^i=x^i(\xi)} = \\ &= \widehat{\omega}(x(\xi))\widehat{A}_i(x(\xi))\widehat{\omega}^{-1}(x(\xi)) - \frac{\partial\widehat{\omega}(x)}{\partial x^i}\bigg|_{x^i=x^i(\xi)}\widehat{\omega}^{-1}(x(\xi)).\end{aligned} \tag{192}$$

We obtain after the change of vierbeins

$$\begin{aligned}a'(\xi + d\xi) &= a'(\xi) - \widehat{A}'_i(a(\xi))a'(\xi)\frac{dx^i(\xi)}{d\xi}d\xi = \widehat{\omega}(x(\xi))a(\xi) - \\ &- \left(\widehat{\omega}(x(\xi))\widehat{A}_i(x(\xi))\widehat{\omega}^{-1}(x(\xi)) - \frac{\partial\widehat{\omega}(x)}{\partial x^i}\bigg|_{x^i=x^i(\xi)}\widehat{\omega}^{-1}(x(\xi))\right)\omega(x(\xi))a(\xi)\frac{dx^i}{d\xi}d\xi = \\ &= \widehat{\omega}(x(\xi))a(\xi) + \frac{\partial\widehat{\omega}(x)}{\partial x^i}\bigg|_{x^i=x^i(\xi)}\frac{dx^i}{d\xi}d\xi a(\xi) - \omega(x(\xi))\widehat{A}_i(x(\xi))a(\xi)\frac{dx^i}{d\xi}d\xi = \\ &= \widehat{\omega}(x(\xi + d\xi))a(\xi) - \omega(x(\xi))\widehat{A}_i(x(\xi))a(\xi)\frac{dx^i}{d\xi}d\xi.\end{aligned} \tag{193}$$

At the necessary 1st order in the $d\xi$ we have

$$a'(\xi + d\xi) = \widehat{\omega}(x(\xi + d\xi))\left(a(\xi) - \widehat{A}_i(x(\xi))a(\xi)\frac{dx^i}{d\xi}d\xi\right) = \widehat{\omega}(x(\xi + d\xi))a(\xi + d\xi). \tag{194}$$

Thus we get the vector form of the transformation:

$$a'(\xi + d\xi) = \widehat{\omega}(x(\xi + d\xi))a(\xi + d\xi). \tag{195}$$

Let us divide the contour Γ in infinitesimally small intervals and, repeating the parallel transportation along the one interval infinitely many times, define the vector $a(\xi)$ at all points of the contour, starting from a given value of this vector at one point. The obtained vector function $a(\xi) \equiv a^a(\xi)$, in accordance with the (190), satisfies at every point of the contour the following condition:

$$a(\xi + d\xi) - a(\xi) + \widehat{A}_i(x(\xi))\widehat{a}(\xi)\frac{dx^i}{d\xi}d\xi = 0, \tag{196}$$

or

$$\left(\frac{da(\xi)}{d\xi} + \widehat{A}_i(x(\xi))\widehat{a}(\xi)\frac{dx^i}{d\xi}\right) d\xi = 0. \tag{197}$$

Due to the arbitrariness of the $d\xi$ we get the differential equation

$$\frac{da(\xi)}{d\xi} + \widehat{A}_i(x(\xi))a(\xi)\frac{dx^i}{d\xi} = 0, \tag{198}$$

or

$$\frac{da^a(\xi)}{d\xi} + \widehat{A}_i{}^a{}_b(x(\xi))a^b(\xi)\frac{dx^i}{d\xi} = 0. \tag{199}$$

This equation is called the equation of parallel transportation of the vector along the curve.

If two points $x_{(1)}$ and $x_{(2)}$ are connected by two different curves, then the parallel transportation from the point $x_{(1)}$ to the point $x_{(2)}$ can give different results. In particular, the vector can change after the parallel transportation along the closed contour to the initial point.

The equation (198) is linear and therefore its solution can be written in the following form:

$$a(\xi) = \widehat{W}(\xi, \xi_1, \Gamma)a(\xi_1), \tag{200}$$

where the $a(\xi_1)$ is the given value of the vector a at the initial point $x(\xi_1)$ of the contour Γ, and the $\widehat{W}(\xi, \xi_1, \Gamma)$ is a matrix depending on the contour Γ, its initial point ξ_1 and the point ξ, to which the vector a is transported. Substituting the (200) into the (198), we see that the matrix $\widehat{W}(\xi, \xi_1, \Gamma)$ satisfies the equation

$$\frac{d\widehat{W}(\xi, \xi_1, \Gamma)}{d\xi} = -\widehat{A}_i(x(\xi))\widehat{W}(\xi, \xi_1, \Gamma)\frac{dx^i(\xi)}{d\xi} \tag{201}$$

at the initial condition

$$\widehat{W}(\xi, \xi_1, \Gamma) = \widehat{I}, \tag{202}$$

where the $\widehat{I}$ is the unit matrix.

The equation (201) has the same form as the Schroedinger equation in quantum mechanics. But we have instead of the time the parameter ξ, and the quantity $-\widehat{A}_i(x(\xi))\frac{dx^i(\xi)}{d\xi}$ instead of the $-iH$.

Therefore the solution of the equation (201) can be, as in the case of Schroedinger equation, written formally as an ordered exponent of the integral over the points of the contour:

$$\widehat{W}(\xi, \xi_1, \Gamma) = P^{\xi \longleftrightarrow \xi_1} \exp\left(- \int\limits_{\xi_1}^{\xi} d\tilde{\xi} \frac{dx^i(\tilde{\xi})}{d\tilde{\xi}} \widehat{A}_i(x(\tilde{\xi})) \right). \tag{203}$$

Here the symbol $P^{\xi \longleftrightarrow \xi_1}$ means the ordering of quantities, depending on points of the contour Γ, from the ξ_1 on the right to the ξ on the left (analogously to the ordering in time in the case of Schroedinger equation). The integral (203) is the analog of the Wilson-Polyakov contour integral in the gauge field theory.

Under the $SO(3)$ – transformations of the vierbein frame,

$$e'^a_i = \omega^a{}_b e^b_i \tag{204}$$

the equality

$$a(\xi) = \widehat{W}(\xi, \xi_1, \Gamma) a(\xi_1). \tag{205}$$

turns into the

$$a'(\xi) = \widehat{W}'(\xi, \xi_1, \Gamma) a'(\xi_1), \tag{206}$$

where

$$a'(\xi) = \widehat{\omega}(x(\xi)) a(\xi), \qquad a'(\xi_1) = \widehat{\omega}(x(\xi_1)) a(\xi_1), \tag{207}$$

so that

$$\widehat{\omega}(x(\xi)) a(\xi) = \widehat{W}'(\xi, \xi_1, \Gamma) \widehat{\omega}(x(\xi_1)) a(\xi_1), \tag{208}$$

or

$$a(\xi) = \widehat{\omega}^{-1}(x(\xi)) \widehat{W}'(\xi, \xi_1, \Gamma) \widehat{\omega}(x(\xi_1)) a(\xi_1). \tag{209}$$

Comparing this with the (205), we find the transformation law of the matrix $\widehat{W}(\xi, \xi_1, \Gamma)$ under the change of the vierbein frame:

$$W(\xi, \xi_1, \Gamma) = \widehat{\omega}^{-1}(x(\xi)) \widehat{W}'(\xi, \xi_1, \Gamma) \widehat{\omega}(x(\xi_1)), \tag{210}$$

or

$$\widehat{W}'(\xi, \xi_1, \Gamma) = \widehat{\omega}(x(\xi)) \widehat{W}(\xi, \xi_1, \Gamma) \widehat{\omega}^{-1}(x(\xi_1)). \tag{211}$$

Let now the contour Γ to be closed (be a loop), so that

$$x(\xi_2) = x(\xi_1), \tag{212}$$

where the $x(\xi_2)$ is the final point of the contour. Then

$$\widehat{W}'(\xi_2, \xi_1, \Gamma) = \widehat{\omega}(x(\xi_2)) \widehat{W}(\xi_2, \xi_1, \Gamma) \omega^{-1}(x(\xi_1)) = \widehat{\omega}(x(\xi_1)) \widehat{W}(\xi_2, \xi_1, \Gamma) \omega^{-1}(x(\xi_1)) \tag{213}$$

and

$$\mathbf{tr}W'(\xi_2,\xi_1,\Gamma)=\mathbf{tr}W(\xi_2,\xi_1,\Gamma), \tag{214}$$

where the $\mathbf{tr}\widehat{W}$ is the trace of the matrix $\widehat{W}$.

Thus, the trace of the contour integral over the points of closed contour is invariant under the $SO(3)$ – transformations of vierbeins. By the vector character of the quantity ${A_i}^a{}_b$ w. r. t. 3-dimensional coordinate transformation the trace of such a contour integral is also invariant under these coordinate transformations, if the contour Γ as a geometrical object is not displaced. But if the coordinate frame carries the contour with itself under the transformation, so that the equation of the contour in new coordinates remains of the same form as in initial coordinates, then the trace of the corresponding integral changes.

New connection ${B_i}^a{}_b$ transforms under the change of vierbein and coordinate frames in the same way as the ${A_i}^a{}_b$. So the contour integral constructed from the connection ${B_i}^a{}_b$,

$$W(\xi,\xi_1,\Gamma,[B])=P^{\xi\longleftrightarrow\xi_1}\exp\left(-\int\limits_{\xi_1}^{\xi}d\xi\frac{dx^i(\xi)}{d\xi}\widehat{B}_i(x(\xi))\right) \tag{215}$$

has the same properties as the contour integral (203) with the connection ${A_i}^a{}_b$. In particular, the trace of the integral (218) taken over the closed contour Γ,

$$\mathbf{tr}W(\xi_2,\xi_1,\Gamma,[B])\quad\text{at}\quad x^i(\xi_1)=x^i(\xi_2) \tag{216}$$

is invariant under the transformation of the 3-dimensional vierbein frame and also 3-dimensional coordinate frame, if the contour is not displaced as a geometrical object at the change of coordinates. Here as in the case of the gauge field, the components of the connection ${B_i}^a{}_b$ are generalized canonical coordinates (after the canonical transformation (170), (171), while the connection ${A_i}^a{}_b$ was a complicated function (146) of the canonical coordinates Q^i_a before the canonical transformation.

Thus, one can easily construct from the canonical variables ${B_i}^a{}_b$ any number of quantities with the above mentioned invariance properties. This is a basic point of the "loop quantum gravity".

In terms of the variables B^a_i, Π^i_a the canonical form of the action is as follows:

$$S^{\mathrm{rep}(B,\Pi)}_{(1)}=\int d^4x\left(\Pi^i_a\partial_0B^a_i-N\mathcal{H}_0-N^i\mathcal{H}_i-\lambda^a\Phi^a\right), \tag{217}$$

where the $\mathcal{H}_0$, $\mathcal{H}_i$ and Φ^a are obtained from the same quantities, defined by the equalities (103), (102), (80), with the help of the substitution

$$Q^i_a=-\frac{1}{b}\Pi^i_a, \tag{218}$$

$$\mathcal{P}^a_i=bB^a_i-bA^a_i\big|_{Q^i_a=-\frac{1}{b}\Pi^i_a}, \tag{219}$$

that follows from the (170), (171). Here, by (146) we have

$$A^c_i\big|_{Q^i_a=-\frac{1}{b}\Pi^i_a}=\varepsilon^{cab}\left(\Pi^a_i\Pi^k_b\Pi^d_l\partial_k\Pi^l_d+\Pi^l_a\Pi^k_b\Pi^d_i\partial_k\Pi^d_l+\Pi^k_b\partial_k\Pi^a_i+\Pi^a_k\partial_i\Pi^k_b\right), \tag{220}$$

where the Π_i^a is defined by the equality

$$\Pi_a^k \Pi_i^a = \delta_i^k \tag{221}$$

(Π_i^a is not obtained from the Π_a^k by the lowering of the index with the β_{ik}).

Let us express explicitly the constraints Φ^a, $\mathcal{H}_0$, $\mathcal{H}_i$ in terms of the B_i^a and Π_a^i. By (80), (219), (218)

$$\Phi^a = \varepsilon^{abc} Q^{ib} \mathcal{P}_i^c = -\varepsilon^{abc} \Pi_b^i B_i^c + \varepsilon^{abc} \Pi_b^i A_i^c. \tag{222}$$

By (220) we have

$$\begin{aligned}
\varepsilon^{abc}\Pi_b^i A_i^c &= \frac{1}{2}\varepsilon^{abc}\Pi_b^i \varepsilon^{cfg}\left(\Pi_i^f \Pi_g^k \Pi_l^d \partial_k \Pi_d^l + \Pi_f^l \Pi_g^k \Pi_i^d \partial_k \Pi_l^d + \Pi_g^k \partial_k \Pi_i^f + \Pi_k^f \partial_i \Pi_g^k\right) = \\
&= \frac{1}{2}\left(\delta^{af}\delta^{bg} - \delta^{ag}\delta^{bf}\right)\Pi_b^i \left(\Pi_i^f \Pi_g^k \Pi_l^d \partial_k \Pi_d^l + \Pi_f^l \Pi_g^k \Pi_i^d \partial_k \Pi_l^d + \Pi_g^k \partial_k \Pi_i^f + \Pi_k^f \partial_i \Pi_g^k\right) = \\
&= \frac{1}{2}\Pi_b^i \big(\Pi_i^a \Pi_b^k \Pi_l^d \partial_k \Pi_d^l + \Pi_a^l \Pi_b^k \Pi_i^d \partial_k \Pi_l^d + \Pi_b^k \partial_k \Pi_i^a + \Pi_k^a \partial_i \Pi_b^k - \\
&\qquad - \Pi_i^b \Pi_a^k \Pi_l^d \partial_k \Pi_d^l - \Pi_b^l \Pi_a^k \Pi_i^d \partial_k \Pi_l^d - \Pi_a^k \partial_k \Pi_i^b - \Pi_k^b \partial_i \Pi_a^k\big) = \\
&= \frac{1}{2}\big(\Pi_a^k \Pi_l^d \partial_k \Pi_d^l + \Pi_a^l \Pi_b^k \partial_k \Pi_l^b + \Pi_b^k \Pi_b^i \partial_k \Pi_i^a + \Pi_k^a \Pi_b^i \partial_i \Pi_b^k - \\
&\qquad - 3\Pi_a^k \Pi_l^d \partial_k \Pi_d^l - \Pi_b^l \Pi_a^k \partial_k \Pi_l^b - \Pi_a^k \Pi_b^i \partial_k \Pi_i^b - \partial_k \Pi_a^k\big).
\end{aligned} \tag{223}$$

Due to

$$\Pi_l^d \partial_k \Pi_d^l = -\Pi_d^l \partial_k \Pi_l^d, \qquad \Pi_k^a \Pi_b^i \partial_i \Pi_b^k = -(\partial_i \Pi_k^a)\Pi_b^i \Pi_b^k, \tag{224}$$

some terms are canceled. Beside of that,

$$\Pi_a^l \Pi_b^k \partial_k \Pi_l^b = -(\partial_k \Pi_a^l)\Pi_b^k \Pi_l^b = -\partial_l \Pi_a^l. \tag{225}$$

Therefore,

$$\varepsilon^{abc}\Pi_b^i A_i^c = -\partial_l \Pi_a^l. \tag{226}$$

By (222), (226)

$$\Phi^a = -\partial_k \Pi_a^k - \varepsilon^{abc}\Pi_b^i B_i^c = -\left(\partial_k \Pi_a^k - \Pi_b^k B_k{}^b{}_a\right), \tag{227}$$

where

$$B_k{}^b{}_a = \varepsilon^{bac} B_i^c. \tag{228}$$

Let us represent this in a slightly different form. Let $\Pi = \det \Pi_a^i$. By (218), (60)

$$\beta = Q \equiv \det Q_a^i = -b^{-3}\Pi. \tag{229}$$

and

$$e_a^i = \frac{Q_a^i}{\sqrt{Q}} = -\frac{1}{b}\frac{\Pi_a^i}{\sqrt{-b^{-3}\Pi}} = -b^{\frac{1}{2}}\frac{\Pi_a^i}{\sqrt{-\Pi}}. \tag{230}$$

Hence the $\Pi^i_a/\sqrt{-\Pi}$ is, as the e^i_a, a stable 3-dimensional tensor. One can define the 3-dimensional covariant derivative of that tensor with the vierbein connection $B_i{}^a{}_b$, denoting it by $\overset{3B}{\nabla}_k\left(\frac{\Pi^i_a}{\sqrt{-\Pi}}\right)$. Then we write

$$\sqrt{-\Pi}\overset{3B}{\nabla}\left(\frac{\Pi^i_a}{\sqrt{-\Pi}}\right)=\sqrt{-\Pi}\left(\partial_i\left(\frac{\Pi^i_a}{\sqrt{-\Pi}}\right)+\overset{3}{\Gamma}{}^i_{ik}\frac{\Pi^k_a}{\sqrt{-\Pi}}-\frac{\Pi^i_a}{\sqrt{-\Pi}}B_i{}^b{}_a\right)=$$
$$=\partial_i\Pi^i_a+\sqrt{-\Pi}\left(\partial_i\frac{1}{\sqrt{-\Pi}}\right)\Pi^i_a+\overset{3}{\Gamma}{}^i_{ik}\Pi^i_a-\Pi^i_bB_i{}^b{}_a. \quad (231)$$

However, by the (229)

$$\sqrt{-\Pi}\left(\partial_i\frac{1}{\sqrt{-\Pi}}\right)\Pi^i_a=\sqrt{-\beta}\left(\partial_i\frac{1}{\sqrt{\beta}}\right)\Pi^i_a=-\frac{1}{\sqrt{-\beta}}\left(\partial_i\sqrt{\beta}\right)\Pi^i_a=-\overset{3}{\Gamma}{}^k_{ki}\Pi^i_a. \quad (232)$$

Therefore

$$\sqrt{-\Pi}\overset{3B}{\nabla}_i\left(\frac{\Pi^i_a}{\sqrt{-\Pi}}\right)=\partial_i\Pi^i_a-\Pi^i_bB_i{}^b{}_a, \quad (233)$$

and the constraint Φ^a, (227), can be written in a covariant form

$$\Phi^a=-\sqrt{-\Pi}\overset{3B}{\nabla}_i\left(\frac{\Pi^i_a}{\sqrt{-\Pi}}\right). \quad (234)$$

To transform the $\mathcal{H}_0$ and $\mathcal{H}_i$ let us introduce the quantity

$$\overset{3}{F}{}^{ab}_{ik}(A)=\partial_iA_k{}^{ab}-\partial_kA_iab+A^{ad}_iA^{db}_k-A^{ad}_kA^{db}_i, \quad (235)$$

constructed from the 3-dimensional vierbein connection A^{ab}_k similarly to the field strength tensor construction from vector potentials in nonabelian gauge field theory. The quantity $\overset{3}{F}{}^{ab}_{ik}(A)$ is simply related with the curvature tensor $\overset{3}{R}{}^l{}_{m,ik}$ of the 3-dimensional hypersurface $x^0=const$. Indeed, the 3-dimensional analog of the equality (22) has the form

$$A^{ab}_i=A_i{}^a{}_b=e^\alpha_l\overset{3}{\Gamma}{}^l_{im}e^m_b+e^a_l\partial_ie^l_b. \quad (236)$$

Substituting the (236) into the (235), we find that

$$\overset{3}{F}{}^{ab}_{ik}(A)=\partial_i\left(e^\alpha_l\overset{3}{\Gamma}{}^l_{km}e^m_b+e^a_l\partial_ke^l_b\right)+$$
$$+\left(e^\alpha_l\overset{3}{\Gamma}{}^l_{im}e^m_d+e^a_l\partial_ie^l_d\right)\left(e^d_n\overset{3}{\Gamma}{}^n_{kq}e^q_b+e^q_n\partial_ke^n_b\right)-(i\longleftrightarrow k), \quad (237)$$

where the $(i\longleftrightarrow k)$ denotes a quantity obtained from a given one by the exchange $i\longrightarrow k$, $k\longrightarrow i$. Further,

$$\overset{3}{F}{}^{ab}_{ik}(A)=e^a_l\left(\partial_i\overset{3}{\Gamma}{}^l_{km}+\overset{3}{\Gamma}{}^l_{im}\overset{3}{\Gamma}{}^m_{kq}\right)e^q_b+(\partial_ie^a_l)\overset{3}{\Gamma}{}^l_{km}e^m_b+e^a_l\overset{3}{\Gamma}{}^l_{km}\partial_ie^m_b+(\partial_ie^a_l)\partial_ke^l_b+$$
$$+e^a_l\partial_i\partial_ke^l_b+e^a_l\overset{3}{\Gamma}{}^l_{im}\partial_ke^m_b-(\partial_ie^a_l)\overset{3}{\Gamma}{}^l_{kq}e^q_b-(\partial_ie^a_l)(\partial_ke^l_b)-(i\longleftrightarrow k). \quad (238)$$

Some terms here are canceled each with other or with the $(i\longleftrightarrow k)$ because in the sum they are symmetric w. r. t. the exchange $i\longrightarrow k$, $k\longrightarrow i$. Therefore

$$\overset{3}{F}{}^{ab}_{ik}(A)=e^a_l\left(\partial_i\overset{3}{\Gamma}{}^l_{km}+\overset{3}{\Gamma}{}^l_{im}\overset{3}{\Gamma}{}^m_{kq}-(i\longleftrightarrow k)\right)e^q_b. \quad (239)$$

At the same time,

$$\overset{3}{R}{}^{l}{}_{m,ik} = \partial_i \overset{3}{\Gamma}{}^{l}_{km} - \partial_k \overset{3}{\Gamma}{}^{l}_{im} + \overset{3}{\Gamma}{}^{l}_{iq} \overset{3}{\Gamma}{}^{q}_{km} - \overset{3}{\Gamma}{}^{l}_{kq} \overset{3}{\Gamma}{}^{q}_{im}, \tag{240}$$

so that

$$\overset{3}{F}{}^{ab}_{ik}(A) = e^a_l \overset{3}{R}{}^{l}{}_{m,ik} e^m_b, \tag{241}$$

i. e. the strength $\overset{3}{F}{}^{ab}_{ik}(A)$ is the 3-dimensional curvature tensor, related in two indices to the vierbein, with $\overset{3}{F}{}^{ab}_{ik}(A) = -\overset{3}{F}{}^{ba}_{ik}(A)$.

Let us define the quantity $\overset{3}{F}{}^{a}_{ik}(A)$ by

$$\overset{3}{F}{}^{ab}_{ik}(A) = \varepsilon^{abc} \overset{3}{F}{}^{c}_{ik}(A). \tag{242}$$

From the equalities (235), (241) and $A_i{}^{ab} = \varepsilon^{abc} A^c_i$ we get

$$\overset{3}{F}{}^{c}_{ik}(A) = \partial_i A^c_k - \partial_k A^c_i - \varepsilon^{cab} A^a_i A^b_k = \frac{1}{2} \varepsilon^{cab} e^l_a e^m_b \overset{3}{R}_{lm,ik}, \tag{243}$$

where $\overset{3}{R}_{lm,ik} = \beta_{ln} \overset{3}{R}{}^{n}{}_{m,ik}$ with

$$\overset{3}{R}_{lm,ik} = \overset{3}{R}_{ik,lm} = -\overset{3}{R}_{ki,lm} = -\overset{3}{R}_{ik,ml}. \tag{244}$$

Together with the $\overset{3}{F}{}^{ab}_{ik}(A)$, $\overset{3}{F}{}^{c}_{ik}(A)$ let us introduce the quantities $\overset{3}{F}{}^{ab}_{ik}(B)$, $\overset{3}{F}{}^{c}_{ik}(B)$, constructed from the connection $B_i{}^a{}_b$ in the same way as the $\overset{3}{F}{}^{ab}_{ik}(A)$, $\overset{3}{F}{}^{c}_{ik}(A)$ are constructed from the $A_i{}^a{}_b$:

$$\overset{3}{F}{}^{ab}_{ik}(B) = \overset{3}{F}{}^{ab}_{ik}(A)\Big|_{A^{ab}_i \longrightarrow B^{ab}_i}, \tag{245}$$

$$\overset{3}{F}{}^{c}_{ik}(B) = \overset{3}{F}{}^{c}_{ik}(A)\Big|_{A^{a}_i \longrightarrow B^{a}_i}, \tag{246}$$

Let us return to the constraints $\mathcal{H}_i$. By (102)

$$\mathcal{H}_i = Q^k_a \left(\overset{3}{\nabla}_k \mathcal{P}^a_i - \overset{3}{\nabla}_i \mathcal{P}^a_k \right). \tag{247}$$

On the other side, by (243), (246), (170)

$$\begin{aligned}
&\overset{3}{F}{}^{c}_{ik}(B) = \partial_i B^c_k - \partial_k B^c_i - \varepsilon^{cab} B^a_i B^b_k = \\
&\quad = \partial_i \left(A^c_k + \frac{1}{b} \mathcal{P}^c_k \right) - \partial_k \left(A^c_k + \frac{1}{b} \mathcal{P}^c_i \right) - \varepsilon^{cab} \left(A^a_i + \frac{1}{b} \mathcal{P}^a_i \right) \left(A^b_k + \frac{1}{b} \mathcal{P}^b_k \right) = \\
&= \overset{3}{F}{}^{c}_{ik}(A) + \\
&+ \frac{1}{b} \left(\left(\partial_i \mathcal{P}^c_k - \varepsilon^{cab} A^a_i \mathcal{P}^b_k - \overset{3}{\Gamma}{}^{l}_{ik} \mathcal{P}^c_l \right) - \left(\partial_k \mathcal{P}^c_i - \varepsilon^{cab} A^a_k \mathcal{P}^b_i - \overset{3}{\Gamma}{}^{l}_{ki} \mathcal{P}^c_l \right) \right) - \frac{1}{b^2} \varepsilon^{cab} \mathcal{P}^a_i \mathcal{P}^b_k,
\end{aligned} \tag{248}$$

where we have added the equal to zero quantity

$$\frac{1}{b}\left(-\overset{3}{\Gamma}{}^l_{ik}\mathcal{P}^c_l-(-\overset{3}{\Gamma}{}^l_{ik}\mathcal{P}^c_l)\right). \tag{249}$$

Or,

$$\begin{aligned}\overset{3}{F}{}^c_{ik}(B)=\overset{3}{F}{}^c_{ik}(A)+\frac{1}{b}\left(\overset{3}{\nabla}_i\mathcal{P}^a_k-\overset{3}{\nabla}_k\mathcal{P}^a_i\right)-\frac{1}{b^2}\varepsilon^{cab}\mathcal{P}^a_i\mathcal{P}^b_k=\\=\frac{1}{2}\varepsilon^{cab}e^l_a e^m_b\overset{3}{R}_{ik,lm}+\frac{1}{b}\left(\overset{3}{\nabla}_i\mathcal{P}^a_k-\overset{3}{\nabla}_k\mathcal{P}^a_i\right)-\frac{1}{b^2}\varepsilon^{cab}\mathcal{P}^a_i\mathcal{P}^b_k,\end{aligned} \tag{250}$$

where we have taken into account the (243).

Hence,

$$\Pi^i_c\overset{3}{F}{}^c_{ik}(B)=\frac{1}{2}\Pi^i_c\varepsilon^{cab}e^l_a e^m_b\overset{3}{R}_{ik,lm}+\frac{1}{b}\Pi^i_c\left(\overset{3}{\nabla}_i\mathcal{P}^c_k-\overset{3}{\nabla}_k\mathcal{P}^c_i\right)-\frac{1}{b^2}\Pi^i_c\varepsilon^{cab}\mathcal{P}^a_i\mathcal{P}^b_k. \tag{251}$$

By (171) $\Pi^i_a=-bQ^i_a$, so that with the equality (247) we obtain

$$\frac{1}{b}\Pi^i_c\left(\overset{3}{\nabla}_i\mathcal{P}^c_k-\overset{3}{\nabla}_k\mathcal{P}^c_i\right)=-Q^i_a\left(\overset{3}{\nabla}_i\mathcal{P}^a_k-\overset{3}{\nabla}_k\mathcal{P}^a_i\right)=-\mathcal{H}_k. \tag{252}$$

Further, the tensor $\overset{3}{R}_{ik,lm}$ satisfies the identity

$$\varepsilon^{ikl}\overset{3}{R}_{ik,lm}=0, \tag{253}$$

which is a 3-dimensional analog of the 4-dimensional identity

$$\varepsilon^{\mu\alpha\beta\gamma}R_{\alpha\beta,\gamma\delta}=0. \tag{254}$$

Therefore, taking into account the equality

$$\Pi^i_a=-bQ^i_a=-bee^i_a, \tag{255}$$

where $e=\det e^a_i=(\det e^i_a)^{-1}$, we find that

$$\Pi^i_c\varepsilon^{cab}e^l_a e^m_b\overset{3}{R}_{ik,lm}=-bee^i_c e^l_a e^m_b\varepsilon^{cab}\overset{3}{R}_{ik,lm}=-b\varepsilon^{ilm}\overset{3}{R}_{ik,lm}=0, \tag{256}$$

because of $\overset{3}{R}_{ik,lm}=\overset{3}{R}_{lm,ik}$.

Finally, by (222) and (255) we have

$$-\frac{1}{b^2}\Pi^i_c\varepsilon^{cab}\mathcal{P}^a_i\mathcal{P}^b_k=\frac{1}{b}Q^i_c\mathcal{P}^a_i\varepsilon^{cab}\mathcal{P}^b_k=\frac{1}{b}\Phi^b\mathcal{P}^b_k. \tag{257}$$

In virtue of the (252), (256), (257) the equality (251) takes the form

$$\Pi^i_c\overset{3}{F}{}^c_{ik}(B)=-\mathcal{H}_k+\frac{1}{b}\mathcal{P}^b_k\Phi^b, \tag{258}$$

and, hence,

$$\mathcal{H}_k=-\Pi^i_c\overset{3}{F}{}^c_{ik}(B)+\frac{1}{b}\mathcal{P}^b_k\Phi^b. \tag{259}$$

Early we have introduced (see (111)) the linear combination of constraints $\mathcal{H}_i + A^c_i\Phi^c$, which generates transformations of 3-dimensional coordinates without a displacement of vierbeins as geometrical objects (in the theory with canonical variables Q^i_a, $\mathcal{P}^a_i$). By (218), (219) we can write this quantity in the form

$$\mathcal{H}_k + A^c_k\Phi^c = -\Pi^i_c \overset{3}{F}{}^c_{ik}(B) + (B^b_k - A^b_k)\Phi^b + A^c_k\Phi^c = -\Pi^i_c \overset{3}{F}{}^c_{ik}(B) + B^b_k\Phi^b. \tag{260}$$

We can use, if it is convenient, instead of the constraint $\mathcal{H}_k$ the constraint

$$\mathcal{H}'_k \equiv \mathcal{H}_k + A^c_k\Phi^c = -\Pi^i_c \overset{3}{F}{}^c_{ik}(B) + B^b_k\Phi^b. \tag{261}$$

Since the transformation $(Q^i_a, \mathcal{P}^a_i) \longrightarrow (B^a_c, \Pi^i_a)$ is canonical the physical sense of the constraints Φ^a (234) and $\mathcal{H}'_k = \mathcal{H}_i + A^c_i\Phi^c$ (261) does not change. The constraints Φ^a generate, as earlier, the $SO(3)$ – transformations of 3-dimensional vierbeins without a change of coordinates, and the constraints $\mathcal{H}'_k = \mathcal{H}_i + A^c_i\Phi^c$, Φ^a generates the transformations of 3-dimensional coordinates without the change of vierbeins as geometrical objects.

Let us return to the constraint $\mathcal{H}_0$. By (250), (171) and the equality $Q^i_a = \beta^{\frac{1}{2}} e^i_a$ we have

$$\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B) = b^2\beta e^i_d e^k_f \varepsilon^{dfc}\left(\frac{1}{2}\varepsilon^{cab}e^l_a e^m_b \overset{3}{R}_{ik,lm} + \frac{1}{b}\left(\overset{3}{\nabla}_i\mathcal{P}^c_k - \overset{3}{\nabla}_k\mathcal{P}^c_i\right) - \frac{1}{b^2}\varepsilon^{cab}\mathcal{P}^a_i\mathcal{P}^b_k\right) =$$
$$= b^2\beta e^i_d e^l_d e^k_f e^m_f \overset{3}{R}_{ik,lm} + 2b\beta\overset{3}{\nabla}_i\left(e^i_d e^k_f \mathcal{P}^c_k\varepsilon^{dfc}\right) - \beta\left(e^i_d e^k_f - e^i_f e^k_d\right)\mathcal{P}^d_i\mathcal{P}^f_k. \tag{262}$$

We have taken into account that

$$\varepsilon^{dfc}\varepsilon^{abc} = \delta^{da}\delta^{fb} - \delta^{db}\delta^{fc}, \qquad \overset{3}{\nabla}_i e^k_a = 0, \qquad \overset{3}{\nabla}_i\varepsilon^{dfc} = 0. \tag{263}$$

Further,

$$b^2\beta e^i_d e^l_d e^k_f e^m_f \overset{3}{R}_{ik,lm} = b^2\beta\beta^{il}\beta^{km}\overset{3}{R}_{ik,lm} = b^2\beta\overset{3}{R}, \tag{264}$$

$$2b\beta\overset{3}{\nabla}_i\left(e^i_d e^k_f\mathcal{P}^c_k\varepsilon^{dfc}\right) = 2b\sqrt{\beta}\partial_i\left(\sqrt{\beta}e^i_d e^k_f\mathcal{P}^c_k\varepsilon^{dfc}\right) =$$
$$= 2b\sqrt{\beta}\partial_i\left(e^i_d\varepsilon^{dfc}Q^k_f\mathcal{P}^c_k\right) = 2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right), \tag{265}$$

$$-\beta\left(e^i_d e^k_f - e^i_f e^k_d\right)\mathcal{P}^d_i\mathcal{P}^f_k = Q^i_f Q^k_d\mathcal{P}^d_i\mathcal{P}^f_k - (Q^i_d\mathcal{P}^d_i)^2 =$$
$$= Q^k_d\mathcal{P}^f_k\left(Q^i_f\mathcal{P}^d_i - Q^i_d\mathcal{P}^f_i\right) + Q^i_d Q^k_d\mathcal{P}^f_i\mathcal{P}^f_k - (Q^i_d\mathcal{P}^d_i)^2 =$$
$$= Q^i_d Q^k_d\mathcal{P}^f_i\mathcal{P}^f_k - (Q^i_d\mathcal{P}^d_i)^2 - Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c. \tag{266}$$

Therefore

$$\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B) = b^2\beta\overset{3}{R} + \left(Q^i_d Q^k_d\mathcal{P}^f_i\mathcal{P}^f_k - (Q^i_d\mathcal{P}^d_i)^2\right) - Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c + 2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right). \tag{267}$$

At the same time, by (103)

$$\mathcal{H}_0 = \frac{1}{4}\left(\frac{2\varkappa}{Q^{\frac{1}{2}}}\right)\left(Q^k_b Q^l_b\mathcal{P}^c_k\mathcal{P}^c_l - (Q^k_b\mathcal{P}^b_k)^2\right) - \left(\frac{Q^{\frac{1}{2}}}{2\varkappa}\right)\left(\overset{3}{R} - 2\Lambda\right). \tag{268}$$

Therefore the equality (267) can be written in two ways: the first one,

$$\begin{aligned}
&\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B) = \\
&= b^2\beta\overset{3}{R}+4\left(\frac{Q^{\frac{1}{2}}}{2\varkappa}\right)\left(\mathcal{H}_0+\left(\frac{Q^{\frac{1}{2}}}{2\varkappa}\right)\left(\overset{3}{R}-2\Lambda\right)\right)-Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c+2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right) = \\
&= 4\left(\frac{Q^{\frac{1}{2}}}{2\varkappa}\right)\left(\mathcal{H}_0+\left(\frac{Q^{\frac{1}{2}}}{2\varkappa}\right)\left(\left(1+\left(\frac{2\varkappa}{2}\right)^2b^2\right)\overset{3}{R}-2\Lambda\right)\right)+ \\
&\qquad -Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c+2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right); \qquad (269)
\end{aligned}$$

the second one, taking into account (218), (219):

$$\begin{aligned}
&\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B) = \\
&= -(2\varkappa)b^2\sqrt{\beta}\mathcal{H}_0+\left(1+\left(\frac{2\varkappa}{2}\right)^2b^2\right)\left(Q^k_bQ^l_b\mathcal{P}^c_k\mathcal{P}^c_l-(Q^k_b\mathcal{P}^b_k)^2\right)+ \\
&\qquad +2b^2\beta\Lambda-Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c+2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right) = \\
&= -(2\varkappa)b^2\sqrt{\beta}\mathcal{H}_0+\left(1+\left(\frac{2\varkappa}{2}\right)^2b^2\right)\left(\Pi^k_b\Pi^l_b(B^c_k-A^c_k)(B^c_l-A^c_l)-\left(\Pi^k_b(B^b_k-A^b_k)\right)^2\right)+ \\
&\qquad +2b^2\beta\Lambda-Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c+2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right), \qquad (270)
\end{aligned}$$

where the A^c_k, Q^k_d, β, e^i_d are to be expressed in terms of the Π^i_a with the help of the (59), (60), (62), (218), (220). Accordingly, the constraint $\mathcal{H}_0$ can be written in two forms:

$$\begin{aligned}
\mathcal{H}_0 = &\frac{1}{4}\left(\frac{2\varkappa}{Q^{\frac{1}{2}}}\right)\left(\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B)+Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c-2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right)\right)- \\
&-\frac{Q^{\frac{1}{2}}}{2\varkappa}\left(\left(1+\left(\frac{2\varkappa}{2}\right)^2b^2\right)\overset{3}{R}-2\Lambda\right) \qquad (271)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}_0 = &\frac{1}{(2\varkappa)b^2\sqrt{\beta}}\Bigg(-\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B)+ \\
&+\left(1+\left(\frac{2\varkappa}{2}\right)^2b^2\right)\left(\Pi^k_b\Pi^l_b(B^c_k-A^c_k)(B^c_l-A^c_l)-\left(\Pi^k_b(B^b_k-A^b_k)\right)^2\right)+ \\
&+2b^2\beta\Lambda-Q^k_d\mathcal{P}^f_k\varepsilon^{dfc}\Phi^c+2b\sqrt{\beta}\partial_i\left(e^i_d\Phi^d\right)\Bigg). \qquad (272)
\end{aligned}$$

The action in the canonical form looks now like the following:

$$S^{\mathrm{rep}(B,\Pi)}_{(1)} = \int d^4x\left(\Pi^i_a\partial_0B^a_i-N\mathcal{H}_0-N^i\mathcal{H}_i-\lambda^a\Phi^a\right). \qquad (273)$$

Substitute here at first the $\mathcal{H}_0$ in the form (271) and replace

$$\int d^4xN\frac{1}{2}(2\varkappa)b\partial_i(e^i_d\Phi^d) \qquad (274)$$

by the

$$\int d^4x \left(-\frac{2\varkappa}{2} b e^i_d (\partial_i N) \Phi^d \right) \tag{275}$$

assuming, for the simplicity, that the universe is closed and throwing out the surface term in the integral[2]. Let us take, further, by (261)

$$N^k \mathcal{H}_k = N^k \mathcal{H}'_k - N^k A^c_k \Phi^c. \tag{276}$$

As a result we find that

$$S^{\mathrm{rep}(B,\Pi)}_{(1)} = \int d^4x \Bigg(\Pi^i_a \partial_o B^a_i - \\ - \left(\frac{1}{4} N \frac{2\varkappa}{Q^{\frac{1}{2}}} \right) \left(\Pi^i_a \Pi^k_b \varepsilon^{abc} \overset{3}{F}{}^c_{ik}(B) - Q \left(\frac{2}{2\varkappa} \right)^2 \left(\left(1 + \left(\frac{2\varkappa}{2} \right)^2 b^2 \right) \overset{3}{R} - 2\Lambda \right) \right) - \\ - N^k \mathcal{H}'_k - \left(\lambda^a + \frac{1}{4} N \frac{2\varkappa}{Q^{\frac{1}{2}}} Q^k_d \mathcal{P}^f_k \varepsilon^{dfa} + \frac{2\varkappa}{2} b e^i_a \partial_i N - N^k A^a_k \right) \Phi^a \Bigg). \tag{277}$$

Now we introduce new lagrange multipliers

$$N' = \frac{1}{4} N \frac{2\varkappa}{\sqrt{Q}}, \tag{278}$$

$$\lambda'^a = \lambda^a + \frac{1}{4} N \frac{(2\varkappa)}{\sqrt{Q}} Q^k_d \mathcal{P}^f_k \varepsilon^{dfa} + \frac{(2\varkappa)}{2} b e^i_a \partial_i N - N^k A^a_k. \tag{279}$$

Since N, λ^a are arbitrary functions, N, λ^a are arbitrary too, and they may be used as lagrange multipliers.

Now we get

$$S^{\mathrm{rep}(B,\Pi)}_{(1)} = \int d^4x \left(\Pi^i_a \partial_0 B^a_i - N' \mathcal{H}'_0 - N^k \mathcal{H}'_k - \lambda'^a \Phi^a \right), \tag{280}$$

where $\mathcal{H}'_k$, ϕ^a are determined by (261), (234) and

$$\mathcal{H}'_0 = \Pi^i_a \Pi^k_b \varepsilon^{abc} \overset{3}{F}{}^c_{ik}(B) + b^{-3} \Pi \left(\frac{2}{(2\varkappa)} \right)^2 \left(\left(1 + \left(\frac{(2\varkappa)}{2} \right)^2 b^2 \right) \overset{3}{R} - 2\Lambda \right). \tag{281}$$

We take into account that

$$\beta = Q = -b^{-3} \Pi \equiv -b^{-3} \det(\Pi^i_a). \tag{282}$$

in accordance with (229). It is presumed, that $\overset{3}{R}$ is expressed in terms of Π^i_a by (218), (65). Lagrange multipliers N' and λ'^a may be also expressed in terms of N, N_k, λ^a, B^c_i, Π^i_c by (218), (219), (146).

Let us represent $S^{\mathrm{rep}(B,\Pi)}_{(1)}$ in the other form with the help of expression (272) for $\mathcal{H}_0$. From this expression and (217), (276) we find

$$S^{\mathrm{rep}(B,\Pi)}_{(1)} = \int d^4x \Bigg(\Pi^i_a \partial_0 B^a_i - \left(-\frac{N}{(2\varkappa) b^2 \sqrt{\beta}} \right) \Bigg(\Pi^i_a \Pi^k_b \varepsilon^{abc} \overset{3}{F}{}^c_{ik}(B) -$$

[2]Even if the space-time is asymptotically flat at free dimensional infinity the surface term can be here neglected because the corresponding expression decrease at $x^i x^i \to \infty$ enough rapidly.

$$-\left(1+\left(\frac{2\varkappa}{2}\right)^2 b^2\right)\left(\left(\Pi^k_b(B^b_k-A^b_k)\right)^2-\Pi^k_b\Pi^l_b(B^c_k-A^c_k)(B^c_l-A^c_l)\right)-2b^2\beta\Lambda\Big)-$$

$$-N^k\mathcal{H}'_k-\left(\lambda^a-\frac{N}{(2\varkappa)b^2\sqrt{\beta}}Q^k_d\mathcal{P}^f_k\varepsilon^{dfa}-\frac{2}{(2\varkappa)b}(\partial_i N)e^i_a-N^kA^a_k\right)\Phi^a\Bigg). \quad (283)$$

If we introduce instead of N, λ^a new lagrange multipliers

$$N''=-\frac{N}{(2\varkappa)b^2\sqrt{\beta}}, \quad (284)$$

$$\lambda''^a=\lambda^a-\frac{N}{(2\varkappa)b^2\sqrt{\beta}}Q^k_d\mathcal{P}^f_k\varepsilon^{dfa}-\frac{2}{(2\varkappa)b}e^i_a\partial_i N-N^kA^a_k, \quad (285)$$

the action (283) will be:

$$S^{\mathrm{rep}(B,\Pi)}_{(1)}=\int d^4x\left(\Pi^i_a\partial_0 B^a_i-N''\mathcal{H}''_0-N^k\mathcal{H}'_k-\lambda''^a\Phi^a\right), \quad (286)$$

where

$$\mathcal{H}''_0=\Pi^i_a\Pi^k_b\varepsilon^{abc}\overset{3}{F}{}^c_{ik}(B)-$$
$$-\left(1+\left(\frac{(2\varkappa)}{2}\right)^2 b^2\right)\left(\left(\Pi^k_b(B^b_k-A^b_k)\right)^2-\Pi^k_b\Pi^l_b(B^c_k-A^c_k)(B^c_l-A^c_l)\right)-\frac{2}{b}\Pi\Lambda. \quad (287)$$

It is supposed here that A^b_k is expressed in terms of $\mathit{\Pi}^k_b$ in accordance with (220) and that $\mathit{\Pi}=\det \mathit{\Pi}^i_a$,

$$\overset{3}{F}{}^c_{ik}(B)=\partial_i B^c_k-\partial_k B^c_i-\varepsilon^{cab}B^a_iB^b_k, \quad (288)$$

and in accordance with (227), (234), (261)

$$\mathcal{H}'_k=-\Pi^i_c\overset{3}{F}{}^c_{ik}(B)+B^b_k\Phi^b, \quad (289)$$

$$\Phi^a=-\sqrt{-\Pi}\overset{3}{\nabla}_i[B]\left(\frac{\Pi^i_a}{\sqrt{-\Pi}}\right)=-\left(\partial_k\Pi^k_a-\Pi^k_bB^b_{ka}\right), \quad (290)$$

where $B^b_{ka}=\varepsilon^{bac}B^c_k$.

One may express lagrange multipliers N'', λ''^a in terms of N, N^k, λ^a, B^c_i, $\mathit{\Pi}^i_c$ with the help of (218), (219), (220) and equalities $\beta=Q=-b^{-3}\mathit{\Pi}$, $e^i_a=Q^{-1/2}Q^i_a$.

It is possible to use the theory, based on any of the two forms for action – (280) or (286).

The variables B^a_i, $\mathit{\Pi}^i_a$ turn, after the quantization, into the operators with commutation relations for fixed x^0 value:

$$\left[B^a_i(x),\Pi^k_b(\widetilde{x})\right]=i\delta^k_i\delta^a_b\delta^3(x-\tilde{x}),\qquad \left[B^a_i(x),B^b_k(\widetilde{x})\right]=0,\qquad \left[\Pi^i_a(x),\Pi^k_b(\widetilde{x})\right]=0. \quad (291)$$

Lagrange multipliers are arbitrary functions and one need seven more subsidiary conditions to fix this arbitrariness. Constraints $\mathcal{H}_0$, $\mathcal{H}_i$, Φ^a are too involved to solve them explicitly. So one has to apply constraints to the physical state vectors:

$$\mathcal{H}''_0|\Psi>=0 \quad (292)$$

or

$$\mathcal{H}'_0|\Psi>=0, \tag{293}$$

$$\mathcal{H}'_i|\Psi>=0, \tag{294}$$

$$\Phi^a|\Psi>=0. \tag{295}$$

It is easy to get state vectors in B_i^a-representation which satisfy the constraints (295). In fact, the constraints Φ^a (295) generate tetrad transformations, and the trace of the closed contour integration of connection B_i^a is invariant under such transformations . Therefore, any function of any number of such traces of different closed paths on the $x^0 = 0$ surface is invariant under $SO(3)$ tetrad transformations and satisfies the constraints (295).

It is slightly more complicated to satisfy constraints $\mathcal{H}'_i$ (294). These constraints generate three-dimensional coordinate transformations, which do not affect tetrad system as geometrical object. Operators $B_i^a(x)$ change not into $B_i'^a(x')$, but into $B_i'^a(x)$. In other words, changing coordinate system carries the integration contour with itself, so that it is not stable geometrical object. Trace of this path integral is not invariant under such transformations. But it is possible, in principe, to construct state vector, which is invariant under such transformation. One need firstly to produce function Ψ of some number of path integral traces and, then, carry out continual integration by all possible transformations of three dimensional coordinate system. It appears a state vector, invariant under constraints $\mathcal{H}'_i$, Φ^a.

If it may be possible to satisfy the constraint (292) (or (293) in other variant), the quantum gravity problem would be solved completely, since generalized hamiltonian is not more than linear combination of the constraints. However, the constraint $\mathcal{H}''_0$ (or $\mathcal{H}'_0$) is much more complicated and one may rely on approximate calculations only. In approximate approach to the problem on the hypersurface $x^0 = 0$ it is usual to introduce lattice and generate closed loops from its edges. It is known, how to get full set of independent state vectors on this lattice, which satisfy constraints (295). Different approximate methods to solve equations (292) (or (293)) are now being developed. This field is known as "loop theory of quantum gravity".

It is ascertain by now, that quantum theory results depend on Barbero-Immirzi parameter b, though in classic the different b theories are connected by canonical transformations and so are equivalent. It is known also that in quantum case the constraint $e^\mu_{(0)} = n^\mu$ results in violation of the local Lorentz invariance of tetrad frame at very small (near Planck length) distances. It does not take place in classic case, where results do not depend on supplementary conditions such as $e^\mu_{(0)} = n^\mu$. Here we deal with quantum anomaly. It does not lessen the value of the theory in itself, since quantum anomaly at very short distances may not contradict the observations. Nevertheless, the other possibility was investigated in PhD thesis by S. Alexandrov [5]. He determined that it is possible to construct theory without violation of the local Lorentz symmetry, but it would be very involved. Therefore, there were no any attempts to develop or to apply this theory.

4 Complex Ashtekar formalism

We will not go here into problems of approximate solutions of the constraint (292) or (293). One can meet it in the articles by Ashtekar, Thiemann and their colleagues. We describe here only the complex Ashtekar formalism, which is of undoubted interest. It may be constructed in the following way.

One can continue the fields B_i^a and Π_a^i to the complex plane, and set

$$b = \pm i \frac{1}{\varkappa}. \tag{296}$$

It is possible to select any sign here. We suppose further that only upper, or only down signs are used. In accordance with (170), (171), (218), (219) the equalities

$$B_i^a = A_i^a \mp i\varkappa \mathcal{P}_i^a, \qquad \Pi_a^i = \mp \frac{i}{\varkappa} Q_a^i, \tag{297}$$

$$Q_a^i = \pm i\varkappa \Pi_a^i, \qquad \mathcal{P}_i^a = \pm i \frac{1}{\varkappa}(B_i^a - A_i^a) \tag{298}$$

take place. At the same time the expressions (281), (287) are simplified abruptly and take the form

$$\mathcal{H}_0' = \mathcal{H}_0'' = \Pi_a^i \Pi_b^k \varepsilon^{abc} \overset{3}{F}{}_{ik}^c(B) \pm 2i\varkappa \Pi \Lambda. \tag{299}$$

The other constraints remain unchanged. Let us note, that under condition $b = \pm i/\varkappa$ an equality $N' = N''$ takes place in accordance with (278), (284).

Taking into account the form of the quantities $\overset{3}{F}{}_{ik}^c(B)$ in (288) we conclude, that all the constraints depend polynomially on the canonical variables B_i^a and Π_a^i. This fact simplifies the theory abruptly. However, in order to return to the real domain (which is physical) we need to impose reality condition onto the solutions

$$B_i^a + B_i^{a*} = 2A_i^a, \tag{300}$$

where A_i^a are expressed by way of (220) in Π. This condition may be considered as second class constraint in complex theory. The existence of this condition is the main problem in present method. The quantity B_i^{a*} in (300) is complex conjugated with B_i^a in classic theory and is Hermitian conjugated with B_i^a in quantum theory.

In view of complexity of the (300), it is currently preferred to construct the loop quantum theory of gravity for a real value of the parameter b than for the complex parameter $b = \pm i/\varkappa$, in spite of the complicated constraint $\mathcal{H}_0'$ (or $\mathcal{H}_0''$) in real variant of the theory.

Ashtekar came to his formalism through four dimensional complex selfdual tetrad connection. Now we turn to this point. Let $C^{AB} = -C^{AB}$ be complex antisymmetric tetrad tensor on the tangent vector bundle on the space-time with the symmetry group $SO(1,3)$. Then the tensor ${}^*C^{AB}$ $(-{}^*C^{AB})$ is named dual (anti-dual) with respect to C^{AB}, if

$$\pm {}^*C^{AB} = -\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}C^{FG}. \tag{301}$$

Tensor a^{AB} is named selfdual (anti-selfdual), if

$$a^{AB} = {}^*a^{AB} = -\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}a^{FG}, \tag{302}$$

$$a^{AB} = -{}^*a^{AB} = \frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}a^{FG}, \tag{303}$$

i. e., if

$$a^{ab} = {}^{*}a^{ab} = -i\varepsilon^{abc}a^{(0)c},$$
$$a^{(0)c} = {}^{*}a^{(0)c} = \frac{i}{2}\varepsilon^{cab}a^{ab}, \tag{304}$$

$$a^{ab} = -{}^{*}a^{ab} = +i\varepsilon^{abc}a^{(0)c},$$
$$a^{(0)c} = -{}^{*}a^{(0)c} = -\frac{i}{2}\varepsilon^{cab}a^{ab}. \tag{305}$$

Any antisymmetric tetrad tensor C^{AB} may be decompose in selfdual and anti-selfdual parts:

$$C^{AB} = C^{(+)AB} + C^{(-)AB}, \tag{306}$$

$$C^{(+)AB} = \frac{1}{2}\left(C^{AB} + {}^{*}C^{AB}\right) = \frac{1}{2}\left(C^{AB} - \frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DERS}C^{RS}\right), \tag{307}$$

$$C^{(-)AB} = \frac{1}{2}\left(C^{AB} - {}^{*}C^{AB}\right) = \frac{1}{2}\left(C^{AB} + \frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DERS}C^{RS}\right), \tag{308}$$

i. e.

$$C^{(\pm)ab} = \frac{1}{2}\left(C^{ab} \mp i\varepsilon^{abc}C^{(0)c}\right),$$
$$C^{(\pm)(0)c} = \frac{1}{2}\left(C^{(0)c} \pm \frac{i}{2}\varepsilon^{cab}C^{ab}\right), \tag{309}$$

and

$$C^{(\pm)ab} = \pm{}^{*}C^{(\pm)ab} = \mp i\varepsilon^{abc}C^{(\pm)(0)c},$$
$$C^{(\pm)(0)c} = \pm{}^{*}C^{(\pm)(0)c} = \pm\frac{i}{2}\varepsilon^{cab}C^{(\pm)ab}. \tag{310}$$

Let us mention that the second equality (310) follows from the first one and vice versa. So there exist only three independence conditions of selfduality. If we had omitted the i in (307) there would be six independent condition and no tensors fulfilling them would exist. So, there do not exist real selfdual tensors.

Working with tensors as $F^{AB}_{\mu\nu}$, which are antisymmetric both in coordinate indices and in tetrad indices (e. g. curvature tensor in tetrad representation), one should bear in mind that along with the conception of selfduality on the tetrad indices A, B one could define by the corresponding manner also the conception of selfduality on the coordinate indices $\mu\nu$. In gauge theories one deals with tensors, selfdual in Lorentz coordinate indices. Here we take an interest in tensors (anti)selfdual in tetrad indices and we shall work with these (anti)selfdual tensors only. In particular, for (anti)selfdual tensor $F^{AB}_{\mu\nu}$ we have

$$F^{AB}_{\mu\nu} = \mp\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}F^{FG}_{\mu\nu}. \tag{311}$$

Let us turn now to the tetrad connection A. Since it is $SO(1,3)$-connection, one has

$$A^{AB}_{\mu} \equiv A^{A}_{\mu\,D}\eta^{DB} = -A^{BA}_{\mu}. \tag{312}$$

The corresponding field strength is

$$F_{\mu\nu}{}^{A}{}_{D}(A) = F_{\mu\nu}^{AB}(A)\eta_{BD} = \partial_\mu A_{\nu}^{A}{}_{D} - \partial_\nu A_{\mu}^{A}{}_{D} + A_{\mu}^{A}{}_{E}A_{\nu}^{E}{}_{D} - A_{\nu}^{A}{}_{E}A_{\mu}^{E}{}_{D}, \tag{313}$$

or

$$F_{\mu\nu}^{AB}(A) = \partial_\mu A_{\nu}^{AB} - \partial_\nu A_{\mu}^{AB} + A_{\mu}^{AD}\eta_{DE}A_{\nu}^{EB} - A_{\nu}^{AD}\eta_{DE}A_{\mu}^{EB} \tag{314}$$

and $F_{\mu\nu}^{AB} = -F_{\mu\nu}^{BA}$. When the frame is changed, which changes the vectors refereed to the tetrad basis by the rule

$$a'^{A} = \omega^{A}{}_{B}a^{B}, \tag{315}$$

where $\omega^{A}{}_{B}$ is a Lorentz matrix, such that

$$\omega^{A}{}_{B}\eta^{BD}\omega^{E}{}_{D} = \eta^{AE}, \quad \det(\omega^{A}{}_{B}) = 1, \tag{316}$$

then the connection changes in accordance with the relation

$$A'^{A}_{\mu}{}_{B} = \omega^{A}{}_{D}A_{\mu}^{D}{}_{F}(\omega^{-1})^{F}{}_{B} + \omega^{A}{}_{B}\partial_\mu(\omega^{-1})^{D}{}_{B}, \tag{317}$$

or, in view of (MT86), (MT87)

$$A'^{AB}_{\mu} = \omega^{A}{}_{D}\omega^{B}{}_{E}A_{\mu}^{DE} + \omega^{A}{}_{D}\eta^{DE}\partial_\mu\omega^{B}{}_{E}, \tag{318}$$

and the corresponding field strength changes as

$$F_{\mu\nu}^{AB}(A') = \omega^{A}{}_{D}\omega^{B}{}_{E}F_{\mu\nu}^{DE}(A). \tag{319}$$

Though the transformation (318) is not tensorial, we can at every frame find (anti)dual quantity with respect to A, if we set

$$\pm{}^{*}A_{\mu}^{AB} = \mp\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFH}A^{FH}. \tag{320}$$

Furthermore,

$$\pm{}^{*}F_{\mu\nu}^{AB}(A) = \mp\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFH}F_{\mu\nu}^{AB}. \tag{321}$$

The ${}^{*}F_{\mu\nu}^{AB}(A)$ is a tensor as is the case with $F_{\mu\nu}^{AB}(A)$. So

$${}^{*}F_{\mu\nu}^{AB}(A') = \omega^{A}{}_{D}\omega^{B}{}_{E}{}^{*}F_{\mu\nu}^{DE}(A). \tag{322}$$

But ${}^{*}A_{\mu}^{AB}$ transforms by the rule

$${}^{*}A'^{AB}_{\mu} = \omega^{A}{}_{D}\omega^{B}{}_{E}{}^{*}A_{\mu}^{DE} - \frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\omega^{F}{}_{H}\eta^{HL}\partial_\mu\omega^{G}{}_{L}, \tag{323}$$

which differs from (318). Then, if we construct $F_{\mu\nu}^{AB}({}^{*}A)$ from ${}^{*}A_{\mu}^{AB}$ just as $F_{\mu\nu}^{AB}(A)$ was constructed from A, we get $F_{\mu\nu}^{AB}({}^{*}A) \neq {}^{*}F_{\mu\nu}^{AB}(A)$ and $F_{\mu\nu}^{AB}({}^{*}A)$ is not a tensor. Let us note, that the first term in the right hand side of (323) looks as tensor, since the first term in the right hand side of (318) is of tensor type and (anti)dual transformation converts the tensor into the tensor. Let us organize (anti)selfdual quantities

$$A_{\mu}^{(\pm)AB} = \frac{1}{2}\left(A_{\mu}^{AB} \pm {}^{*}A_{\mu}^{AB}\right), \tag{324}$$

$$F^{(\pm)AB}_{\mu\nu}(A) = \frac{1}{2}\left(F^{AB}_{\mu\nu}(A) \pm {}^*F^{AB}_{\mu\nu}(A)\right). \tag{325}$$

From here we get

$$A^{AB}_{\mu} = A^{(+)AB}_{\mu} + A^{(-)AB}_{\mu}, \tag{326}$$

$$F^{AB}_{\mu\nu} = F^{(+)AB}_{\mu\nu} + F^{(-)AB}_{\mu\nu}, \tag{327}$$

$${}^*A^{(\pm)AB}_{\mu} = \pm A^{(\pm)AB}_{\mu}, \tag{328}$$

$${}^*F^{(\pm)AB}_{\mu\nu}(A) = \pm F^{(\pm)AB}_{\mu\nu}(A). \tag{329}$$

$F^{(\pm)AB}_{\mu\nu}(A)$ are tensors, so under frame transformation

$$F'^{(\pm)AB}_{\mu\nu}(A) = \omega^A{}_D\omega^B{}_E F^{(\pm)DE}_{\mu\nu}(A), \tag{330}$$

and due to (318), (323)

$$A^{(\pm)\prime AB}_{\mu} = \omega^A{}_D\omega^B{}_E A^{(\pm)DE}_{\mu} + \frac{1}{2}\left(\omega^A{}_D\eta^{DE}\partial_\mu\omega^A{}_E \mp \frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\omega^F{}_H\eta^{HL}\partial_\mu\omega^G{}_L\right). \tag{331}$$

Now we construct $F^{AB}_{\mu\nu}(A^{(\pm)})$ from $A^{(\pm)AB}_{\mu}$ in the same way as $F^{AB}_{\mu\nu}(A)$ is constructed from A^{AB}_{μ}:

$$F^{AB}_{\mu\nu}(A^{(\pm)}) \equiv \partial_\mu A^{(\pm)AB}_{\nu} - \partial_\nu A^{(\pm)AD}_{\mu} + A^{(\pm)AD}_{\mu}\eta_{DE}A^{(\pm)EB}_{\nu} - A^{(\pm)AD}_{\nu}\eta_{DE}A^{(\pm)EB}_{\mu}. \tag{332}$$

The following proposition is valid

$$F^{(\pm)AB}_{\mu\nu}(A) = F^{AB}_{\mu\nu}(A^{(\pm)}), \tag{333}$$

and, in accordance with (330)

$$F^{AB}_{\mu\nu}(A^{(\pm)\prime}) = \omega^A{}_D\omega^B{}_E F^{DE}_{\mu\nu}(A^{(\pm)}). \tag{334}$$

It is true in spite of the fact, that ${}^*F^{AB}_{\mu\nu} \neq F^{AB}_{\mu\nu}({}^*A)$ and the transformation rule for $A^{(\pm)AB}_{\mu}$ (331) differs from the transformation rule for A^{AB}_{μ}.

Let us prove (333) for selfdual tensors (in upper indices). Anti-selfdual case is similar. We have

$$\begin{aligned}
&F^{AB}_{\mu\nu}(A^{(+)}) = \partial_\mu A^{(+)AB}_{\nu} + A^{(+)AD}_{\mu}\eta_{DE}A^{(+)ED}_{\nu} - (\mu \longleftrightarrow \nu) = \\
&= \frac{1}{2}\left(\partial_\mu A^{AB}_{\nu} + \partial_\mu {}^*A^{AB}_{\nu}\right) + \frac{1}{4}\left(A^{AD}_{\mu} + {}^*A^{AD}_{\mu}\right)\eta_{DE}\left(A^{EB}_{\nu} + {}^*A^{EB}_{\nu}\right) - (\mu \longleftrightarrow \nu) = \\
&= \frac{1}{2}\left(\partial_\mu A^{AB}_{\nu} + \frac{1}{2}A^{AD}_{\mu}\eta_{DE}A^{EB}_{\nu} + \frac{1}{2}{}^*A^{AD}_{\mu}\eta_{DE}{}^*A^{ED}_{\nu}\right) + \\
&\qquad + \frac{1}{2}\left(\partial_\mu {}^*A^{AB}_{\nu} + \frac{1}{2}{}^*A^{AD}_{\mu}\eta_{DE}A^{EB}_{\nu} + \frac{1}{2}A^{AD}_{\mu}\eta_{DE}{}^*A^{EB}_{\nu}\right) - (\mu \longleftrightarrow \nu).
\end{aligned} \tag{335}$$

The next equality is valid:

$$\eta^{AF}\varepsilon_{ABDE}\varepsilon_{FGHL} = -\big(\delta_{BG}\delta_{DH}\delta_{EL} + \delta_{BH}\delta_{DL}\delta_{EG} + \delta_{BL}\delta_{DG}\delta_{EH} -$$

$$-\delta_{BG}\delta_{DL}\delta_{EH}-\delta_{BL}\delta_{DH}\delta_{EG}-\delta_{BH}\delta_{DG}\delta_{EL}\Big), \quad (336)$$

and if $a^{HL}=-a^{LH}$, then

$$\eta^{AF}\varepsilon_{ABDE}\varepsilon_{FGHL}a^{HL}=-2\left(\delta_{BG}\delta_{DH}\delta_{EL}+\delta_{BH}\delta_{DL}\delta_{EG}+\delta_{BL}\delta_{DG}\delta_{EH}\right)a^{HL}. \quad (337)$$

Using this fact, we get

$$\begin{aligned}
&{}^*A_\mu^{AD}\eta_{DE}{}^*A_\nu^{EB}-(\mu\longleftrightarrow\nu)=\\
&=-\frac{i}{2}\eta^{AF}\eta^{DG}\varepsilon_{FGHL}A_\mu^{HL}\eta_{DE}\left(-\frac{i}{2}\right)\eta^{EM}\eta^{BN}\varepsilon_{MNPQ}A_\nu^{PQ}-(\mu\longleftrightarrow\nu)=\\
&=-\frac{1}{4}\eta^{GM}\varepsilon_{FGHL}A_\mu^{HL}\varepsilon_{MNPQ}A_\nu^{PQ}\eta^{AF}\eta^{BN}=\frac{1}{4}\eta^{GM}\varepsilon_{GFHL}\varepsilon_{MNPQ}A_\mu^{HL}A_\nu^{PQ}\eta^{AF}\eta^{BN}=\\
&=-\frac{1}{2}\left(\eta_{FN}\eta_{HP}\eta_{LQ}+\eta_{FP}\eta_{HQ}\eta_{LN}+\eta_{FQ}\eta_{HN}\eta_{LP}\right)A_\mu^{HL}A_\nu^{PQ}\eta^{AF}\eta^{BN}=\\
&=-\frac{1}{2}\left(\eta^{AB}A_\mu^{HL}A_\nu^{PQ}\eta_{HP}\eta_{LQ}+A_\mu^{HB}\eta_{HQ}A_\nu^{AQ}+A_\mu^{BL}\eta_{LP}A_\nu^{PA}\right)-(\mu\longleftrightarrow\nu)=\\
&=-\frac{1}{2}\left(-A_\nu^{HB}\eta_{HQ}A_\mu^{AQ}-A_\nu^{BL}\eta_{LP}A_\mu^{PA}\right)-(\mu\longleftrightarrow\nu)=\\
&=\frac{1}{2}\left(A_\mu^{AQ}\eta_{QH}A_\nu^{HB}+A_\mu^{AP}\eta_{PL}A_\nu^{LB}\right)=A_\mu^{AD}\eta_{DE}A_\nu^{EB}. \quad (338)
\end{aligned}$$

It means that

$$\begin{aligned}
&\partial_\mu A_\nu^{AB}+\frac{1}{2}A_\mu^{AD}\eta_{DE}A_\nu^{EB}+\frac{1}{2}{}^*A_\mu^{AD}\eta_{DE}{}^*A_\nu^{ED}-(\mu\longleftrightarrow\nu)=\\
&=\partial_\mu A_\nu^{AB}+A_\mu^{AD}\eta_{DE}A_\nu^{EB}-(\mu\longleftrightarrow\nu)=F_{\mu\nu}^{AB}(A), \quad (339)
\end{aligned}$$

so that in accordance with (335)

$$\begin{aligned}
&F_{\mu\nu}^{AB}(A^{(+)})=\frac{1}{2}F_{\mu\nu}^{AB}(A)+\\
&+\left(\frac{1}{2}\left(\partial_\mu{}^*A_\nu^{AB}+\frac{1}{2}{}^*A_\mu^{AD}\eta_{DE}A_\nu^{EB}+\frac{1}{2}A_\mu^{AD}\eta_{DE}{}^*A_\nu^{EB}\right)-(\mu\longleftrightarrow\nu)\right). \quad (340)
\end{aligned}$$

Further, the equality

$$-\frac{1}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\eta^{FH}\eta^{GL}\varepsilon_{HLMN}=\left(\delta_M^A\delta_N^B-\delta_N^A\delta_M^B\right) \quad (341)$$

is identical. If $a^{AB}=-a^{BA}$, then

$$-\frac{1}{4}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\eta^{FH}\eta^{GL}\varepsilon_{HLMN}a^{MN}=a^{AB}. \quad (342)$$

The expression

$${}^*A_\mu^{AD}\eta_{DE}A_\nu^{EB}+A_\mu^{AD}\eta_{DE}{}^*A_\nu^{EB}-(\mu\longleftrightarrow\nu) \quad (343)$$

is antisymmetric in A, B due to the $(\mu\leftrightarrow\nu)$. So, we have, taking (337) into account,

$$\begin{aligned}
&{}^*A_\mu^{AD}\eta_{DE}A_\nu^{EB}+A_\mu^{AD}\eta_{DE}{}^*A_\nu^{EB}-(\mu\longleftrightarrow\nu)=\\
&=-\frac{1}{4}\eta^{AF}\eta^{BG}\varepsilon_{FGHL}\eta^{HM}\eta^{LN}\varepsilon_{MNQP}\Big(-\frac{i}{2}\eta^{QR}\eta^{DS}\varepsilon_{RSXZ}A_\mu^{XZ}\eta_{DE}A_\nu^{EP}+
\end{aligned}$$

$$+A_\mu^{QD}\eta_{DE}\left(-\frac{i}{2}\right)\eta^{ER}\eta^{PS}\varepsilon_{RSXZ}A_\nu^{XZ}\Bigg)-(\mu\longleftrightarrow\nu)=$$

$$=\frac{i}{8}\eta^{AF}\eta^{BG}\varepsilon_{FGHL}\eta^{HM}\eta^{LN}\Bigg(\eta^{QR}\varepsilon_{QMNP}\varepsilon_{RSXZ}A_\mu^{XZ}A_\nu^{SP}+$$

$$+\eta^{PS}\varepsilon_{PMNQ}\varepsilon_{SRXZ}A_\mu^{QR}A_\nu^{XZ}\Bigg)-(\mu\longleftrightarrow\nu)=$$

$$=-\frac{i}{8}\eta^{AF}\eta^{BG}\varepsilon_{FGHL}\eta^{HM}\eta^{LN}\Big(2\left(\eta_{MS}\eta_{NX}\eta_{PZ}+\eta_{MX}\eta_{NZ}\eta_{PS}+\eta_{MZ}\eta_{PX}\eta_{NS}\right)A_\mu^{XZ}A_\nu^{SP}+$$

$$+2\left(\eta_{MR}\eta_{NX}\eta_{QZ}+\eta_{MX}\eta_{NZ}\eta_{QR}+\eta_{MZ}\eta_{NR}\eta_{QX}\right)A_\mu^{QR}A_\nu^{XZ}-(\mu\longleftrightarrow\nu)\Big)=$$

$$=-\frac{i}{4}\eta^{AF}\eta^{BG}\varepsilon_{FGHL}\big(-A_\nu^{HP}\eta_{PZ}A_\mu^{ZL}+A_\mu^{HX}\eta_{XP}A_\nu^{PL}+$$

$$+A_\mu^{HQ}\eta_{QZ}A_\nu^{ZL}-A_\nu^{HX}\eta_{XQ}A_\mu^{QL}-(\mu\longleftrightarrow\nu)\big)=$$

$$=-\frac{i}{2}\eta^{AF}\eta^{BG}\varepsilon_{FGHL}\left(A_\mu^{HX}\eta_{XP}A_\nu^{PL}-A_\nu^{HP}\eta_{PZ}A_\mu^{ZL}-(\mu\longleftrightarrow\nu)\right)=$$

$$=-i\eta^{AF}\eta^{BG}\varepsilon_{FGHL}A_\mu^{HX}\eta_{XP}A_\nu^{PL}-(\mu\longleftrightarrow\nu),\quad(344)$$

and, further, by (340)

$$\frac{1}{2}\left({}^*A_\nu^{AB}+\frac{1}{2}{}^*A_\mu^{AD}\eta_{DE}A_\nu^{EB}+\frac{1}{2}A_\mu^{AD}\eta_{DE}{}^*A_\nu^{EB}\right)-(\mu\longleftrightarrow\nu)=$$

$$=\frac{1}{2}\left(-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\left(\partial_\mu A_\nu^{FG}+A_\mu^{FH}\eta_{HL}A_\nu^{LG}-(\mu\longleftrightarrow\nu)\right)\right)=$$

$$=\frac{1}{2}\left(-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}F_{\mu\nu}^{FG}(A)\right)=\frac{1}{2}{}^*F_{\mu\nu}^{AB}(A)\quad(345)$$

and

$$F_{\mu\nu}^{AB}(A^{(+)})=\frac{1}{2}\left(F_{\mu\nu}^{AB}(A)+{}^*F_{\mu\nu}^{AB}\right)=F^{(+)AB}(A).\quad(346)$$

The proposition (333) is proved.

So, one may get the selfdual (anti-selfdual) part of the field strength from the selfdual (anti-selfdual) part of the connection dy the same way as usual field strength from the usual connection.

The next proposition is as well true (it is like one which was proved just now, but not the same). Let $a_\mu^{(\pm)AB}$ be any (anti)selfdual quantity with arbitrary transformation rule under the frame transformations and let

$$F_{\mu\nu}^{AB}(a^{(\pm)})=\partial_\mu a_\nu^{(\pm)AB}-\partial_\nu a_\mu^{(\pm)AB}+a_\mu^{(\pm)AD}\eta_{DE}a^{(\pm)EB}-a_\nu^{(\pm)AD}\eta_{DE}a_\mu^{(\pm)EB}.\quad(347)$$

Now we state that $F_{\mu\nu}^{AB}(a^{(\pm)})$ is (anti)selfdual, i. e. if ${}^*a_\mu^{(\pm)AB}=\pm a_\mu^{AB}$, then ${}^*F_{\mu\nu}^{AB}(a^{(\pm)})=\pm F(a^{(\pm)})$.

Note: if $a_\mu^{(\pm)AB}$ does not transforms as (anti)selfdual part of the connection $A_\mu^{(\pm)AB}$, then $F_{\mu\nu}^{AB}(a^{(\pm)})$ is not, in general, a tensor, but it is (anti)selfdual.

Corollary: if $c=\text{const}$, $c\neq 1$, then $F_{\mu\nu}^{AB}(cA^{(\pm)})$ is (anti)selfdual, though it is not a tensor.

We prove the above statement for selfdual case (anti-selfdual case is similar). Let us take into account, that

$$a_\mu^{(+)AB}=-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}a^{(+)FG}.\quad(348)$$

We have

$$-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DBFG}F^{FG}_{\mu\nu}(a^{(+)}) =$$
$$= \partial_\mu\left(-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}a^{(+)FG}_\nu\right) - \partial_\nu\left(-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}a^{(+)FG}_\mu\right) -$$
$$-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\left(a^{(+)FH}_\mu\eta_{HM}a^{(+)MG}_\nu - a^{(+)FH}_\nu\eta_{HM}a^{(+)MG}_\mu\right) =$$
$$= \partial_\mu a^{(+)AB}_\nu - \partial_\nu a^{(+)AB}_\mu -$$
$$-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}\left(-\frac{i}{2}\eta^{FN}\eta^{HP}\varepsilon_{NPRS}a^{(+)RS}_\mu\eta_{HM}a^{(+)MG}_\nu - (\mu \longleftrightarrow \nu)\right) =$$
$$= \partial_\mu a^{(+)AB}_\nu - \partial_\nu a^{(+)AB}_\mu - \frac{1}{4}\left(\eta^{AD}\eta^{BE}\eta^{EN}\varepsilon_{DEFG}\varepsilon_{NMRS}a^{(+)RS}_\mu a^{(+)MG}_\nu - (\mu \longleftrightarrow \nu)\right). \quad (349)$$

Further

$$\eta^{EN}\varepsilon_{FDES}\varepsilon_{NMRS}a^{(+)RS}_\mu =$$
$$= -\left(\eta_{DM}\eta_{ER}\eta_{GS} + \eta_{DR}\eta_{ES}\eta_{GM} + \eta_{DS}\eta_{EM}\eta_{GR} - (R \longleftrightarrow S)\right)a^{(+)RS}_\mu =$$
$$= -2\left(\eta_{DM}\eta_{ER}\eta_{GS} + \eta_{DR}\eta_{ES}\eta_{GM} + \eta_{DS}\eta_{EM}\eta_{GR}\right)a^{(+)RS}_\mu. \quad (350)$$

Then,

$$-\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}F^{FG}_{\mu\nu}(a^{(+)}) = \partial_\mu a^{(+)AB}_\nu - \partial_\nu a^{(+)AB}_\mu +$$
$$+\frac{1}{2}\eta^{AD}\eta^{BE}\left(\left(\eta_{DM}\eta_{ER}\eta_{GS} + \eta_{DR}\eta_{ES}\eta_{GM} + \eta_{DS}\eta_{EM}\eta_{GR}\right)a^{(+)RS}_\mu a^{(+)MG}_\nu - (\mu \longleftrightarrow \nu)\right) =$$
$$= \partial_\mu a^{(+)AB}_\nu - \partial_\nu a^{(+)AB}_\mu +$$
$$+\frac{1}{2}\left(a^{(+)BS}_\mu a^{(+)AG}_\nu\eta_{SG} + a^{(+)AB}_\mu a^{(+)MG}_\nu\eta_{MG} + a^{(+)RA}_\mu a^{(+)BG}_\nu\eta_{RG} - (\mu \longleftrightarrow \nu)\right) =$$
$$= \partial_\mu a^{(+)AB}_\nu - \partial_\nu a^{(+)AB}_\mu + a^{(+)AD}_\mu\eta_{DE}a^{(+)EB}_\nu - a^{(+)AD}_\nu\eta_{DE}a^{(+)EB}_\mu = F^{AB}_{\mu\nu}(a^{(+)}), \quad (351)$$

so,

$$F^{AB}_{\mu\nu}(a^{(+)}) = -\frac{i}{2}\eta^{AB}\eta^{BE}\varepsilon_{DEFG}F^{FG}_{\mu\nu}(a^{(+)}). \quad (352)$$

It may be proved in a similar manner that

$$F^{AB}_{\mu\nu}(a^{(-)}) = \frac{i}{2}\eta^{AB}\eta^{BE}\varepsilon_{DEFG}F^{FG}_{\mu\nu}(a^{(-)}). \quad (353)$$

So, if

$$a^{(\pm)AB}_\mu = \mp\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}a^{(\pm)FG}_\mu, \quad (354)$$

then

$$F^{AB}_{\mu\nu}(a^{(\pm)}) = \mp\frac{i}{2}\eta^{AD}\eta^{BE}\varepsilon_{DEFG}F^{(\pm)FG}_{\mu\nu}(a^{(\pm)}). \quad (355)$$

It means that if $a^{(\pm)AB}_\mu$ is (anti)selfdual, then $F^{AB}_{\mu\nu}(a^{(\pm)})$ is also (anti)selfdual.

Let us consider now expression for the three-dimensional part of the (anti)selfdual field strength with the help of (333), (310) (later we adopt a, b, ... = 1, 2, 3):

$$F^{(\pm)ab}_{\mu\nu}(A) = F^{ab}_{\mu\nu}(A^{(\pm)}) = \partial_\mu A^{(\pm)ab}_\nu + A^{(\pm)aD}_\mu \eta_{DE} A^{(\pm)Eb}_\nu - (\mu \longleftrightarrow \nu) =$$
$$= \partial_\mu A^{(\pm)ab} + A^{(\pm)ac}_\mu A^{(\pm)cb}_\nu + A^{(\pm)a(0)}_\mu \eta_{(0)(0)} A^{(\pm)(0)b}_\nu - (\mu \longleftrightarrow \nu). \quad (356)$$

Same time with the help of (310)

$$A^{(\pm)a(0)}_\mu \eta_{(0)(0)} A^{(\pm)(0)b}_\nu - (\mu \longleftrightarrow \nu) = -A^{(\pm)a(0)}_\mu A^{(\pm)(0)b}_\nu - (\mu \longleftrightarrow \nu) =$$
$$= A^{(\pm)(0)a}_\mu A^{(\pm)(0)b}_\nu - (\mu \longleftrightarrow \nu) = \left(\pm\frac{i}{2}\varepsilon^{acd} A^{(\pm)cd}_\mu\right)\left(\pm\frac{i}{2}\varepsilon^{bfg} A^{(\pm)fg}_\nu\right) - (\mu \longleftrightarrow \nu) =$$
$$= -\frac{1}{4}\varepsilon^{acd} A^{(\pm)cd}_\mu \varepsilon^{bfg} A^{(\pm)fg}_\nu. \quad (357)$$

Since $A^{(\pm)cd}_\mu = -A^{(\pm)dc}_\nu$, then

$$\varepsilon^{acd}\varepsilon^{bfg} A^{(\pm)cd}_\mu = 2\left(\delta^{ab}\delta^{cf}\delta^{dg} + \delta^{af}\delta^{cg}\delta^{db} + \delta^{ag}\delta^{cb}\delta^{df}\right) A^{(\pm)cd}_\mu =$$
$$= 2\left(\delta^{ab} A^{(\pm)fg}_\mu + \delta^{af} A^{(\pm)gb}_\mu + \delta^{ag} A^{(\pm)bf}_\mu\right) \quad (358)$$

and

$$A^{(\pm)(0)a}_\mu A^{(\pm)(0)b}_\nu - (\mu \longleftrightarrow \nu) =$$
$$= -\frac{1}{2}\left(\delta^{ab} A^{(\pm)fg}_\mu + \delta^{af} A^{(\pm)gb}_\mu + \delta^{ag} A^{(\pm)bf}_\mu\right) A^{(\pm)fg}_\nu - (\mu \longleftrightarrow \nu) =$$
$$= -\frac{1}{2}\left(\delta^{ab} A^{(\pm)fg}_\mu A^{(\pm)fg}_\nu + A^{(\pm)gb}_\mu A^{(\pm)ag}_\nu + A^{(\pm)bf}_\mu A^{(\pm)fa}_\nu - (\mu \longleftrightarrow \nu)\right) =$$
$$= -A^{(\pm)gb}_\mu A^{(\pm)ag}_\nu - (\mu \longleftrightarrow \nu) = A^{(\pm)ag}_\mu A^{(\pm)gb}_\nu - (\mu \longleftrightarrow \nu). \quad (359)$$

In view of (356)

$$F^{(\pm)ab}_{\mu\nu}(A) = \partial_\mu A^{(\pm)ab}_\nu - \partial_\nu A^{(\pm)ab}_\mu + 2(A^{(\pm)ac}_\mu A^{(\pm)cb}_\nu - A^{(\pm)ac}_\nu A^{(\pm)cb}_\mu). \quad (360)$$

So,

$$2F^{(\pm)ab}_{\mu\nu}(A) =$$
$$= \partial_\mu(2A^{(\pm)ab}_\nu) - \partial_\nu(2A^{(\pm)ab}_\mu) + (2A^{(\pm)ac}_\mu)(2A^{(\pm)cd}_\nu) - (2A^{(\pm)ac}_\nu)(2A^{(\pm)cb}_\mu) \equiv \overset{3}{F}{}^{ab}_{\mu\nu}(2A^{(\pm)df}_\lambda), \quad (361)$$

where we denote by $\overset{3}{F}{}^{ab}_{\mu\nu}(2A^{(\pm)df}_\lambda)$ the quantity, which is constructed from $2A^{(\pm)df}_i$ in the same way as the three-dimensional (in frame indices) field strength is constructed from $SO(3)$ connection $(2A^{(\pm)df}_\lambda)$ on the vector bundle above the three-dimensional base with $SO(3)$ structure. It is clear, that in arbitrary $SO(1,3)$ frame system we have no frame subsystem, which would be $SO(3)$ frame system above the three-dimensional base. But the quantity

$$2F^{(\pm)ab}_{\mu\nu}(A) = 2F^{ab}_{\mu\nu}(A^{(\pm)}) = \overset{3}{F}{}^{ab}_{\mu\nu}(2A^{(\pm)df}_\lambda) \quad (362)$$

is always so constructed as if we deal with such $SO(3)$ frame subsystem with connection $2A^{(\pm)df}_\lambda$, where index λ runs only three values.

Let us note that in some reviews on the loop gravity the connection between $F^{(\pm)ab}_{\mu\nu}(A)$, $F^{ab}_{\mu\nu}(A^{(\pm)})$ and $\overset{3}{F}{}^{ab}_{\mu\nu}(2A^{(\pm)df}_{\lambda})$ is displayed insufficiently thoroughly.

Let us now turn to the case, when $SO(1,3)$ tetrad system is subject to supplementary condition

$$e^{\mu}_{(0)}(x) = n^{\mu}(x), \tag{363}$$

where $n^{\mu}(x)$ is a unit normal to the $x^0 = const$ hypersurface, which include point x. Earlier during the construction of the gravitational theory in tetrad formalism we all the time supposed, that the condition (363) takes place. Then after (45) $e^{(0)}_i = 0$. This condition must hold under frame transformation, so

$$0 = e'^{(0)}_i = \omega^{(0)}{}_A e^A_i = \omega^{(0)}{}_a e^a_i. \tag{364}$$

Since $\det e^a_i \neq 0$, then

$$\omega^{(0)}{}_a = 0. \tag{365}$$

It follows from

$$\omega^A{}_B \eta^{BD} \omega^E{}_D = \eta^{AE}. \tag{366}$$

and (365) that

$$\omega^{(0)}{}_B \eta^{BD} \omega^a{}_D = \omega^{(0)}{}_{(0)} \eta^{(0)(0)} \omega^a{}_{(0)} = 0, \tag{367}$$

from where

$$\omega^a{}_{(0)} = 0. \tag{368}$$

Furthermore, in accordance with (365), (366), (368)

$$\omega^{(0)}{}_{(0)} \eta^{(0)(0)} \omega^{(0)}{}_{(0)} = \eta^{(0)(0)}, \tag{369}$$

from where $(\omega^{(0)}{}_{(0)})^2 = 1$. We assume, that the direction of e^{μ}_0 does not depend on time, then $\omega^{(0)}{}_{(0)} = 1$.

So, under accepted restrictions on frame system

$$\omega^{(0)}{}_{(0)} = 1, \quad \omega^{(0)}{}_a = 0, \quad \omega^a{}_{(0)} = 0, \tag{370}$$

and the condition (366) turns into

$$\omega^a{}_b \omega^c{}_b = \delta^{ac}. \tag{371}$$

Now three-dimensional part e^i_a of tetrads make up tetrad system on vector $SO(3)$ bundle above the surface $x^0 = \text{const}$, and matrices $\omega^a{}_b$ realize $SO(3)$ rotation of this system.

The transformation rule for the connection

$$A'^{AB}_{\mu} = \omega^A{}_D \omega^B{}_E A^{DE}_{\mu} + \omega^A{}_D \eta^{DE} \partial_{\mu} \omega^B{}_E \tag{372}$$

takes the form

$$A'^{ab}_{\mu} = \omega^a{}_c \omega^b{}_d A^{cd}_{\mu} + \omega^a{}_c \partial_{\mu} \omega^b{}_c, \quad A'^{(0)c}_{\mu} = \omega^c{}_D A^{(0)d}. \tag{373}$$

As it was shown above, three-dimensional part of the connection A_i^{ab} coincides with three-dimensional connection $\overset{3}{A}{}_i^{ab}$, which was built as usual on tetrads e_i^a, formed in three-dimensional system on the surface $x^0 = const$. In accordance with (309)

$$A_\mu^{(\pm)ab} = \frac{1}{2}\left(A_\mu^{ab} \mp i\varepsilon^{abc}A_\mu^{(0)c}\right), \tag{374}$$

$$A_\mu^{(\pm)(0)c} = \frac{1}{2}\left(A_\mu^{(0)c} \pm \frac{i}{2}\varepsilon^{cab}A_\mu^{ab}\right). \tag{375}$$

We find from (373) that transformation rule for $A_\mu^{(\pm)ab}$ takes the form

$$A_\mu^{(\pm)'ab} = \frac{1}{2}\left(\omega^a{}_c\omega^b{}_d A_\mu^{cd} + \omega^a{}_c\partial_\mu\omega^b{}_c \mp i\varepsilon^{abc}\omega^c{}_d A^{(0)d}\right), \tag{376}$$

but

$$\omega^a{}_c\omega^b{}_d\omega^f{}_g\varepsilon^{cdg} = \varepsilon^{abf}, \tag{377}$$

from where in view of (371)

$$\varepsilon^{abc}\omega^c{}_f = \omega^a{}_c\omega^b{}_d\omega\varepsilon^{cdf}. \tag{378}$$

So

$$\varepsilon^{abc}\omega^c{}_d A^{(0)d} = \omega^a{}_c\omega^b{}_d\varepsilon^{cdf}A^{(0)f} \tag{379}$$

and

$$A_\mu^{(\pm)ab} = \frac{1}{2}\left(\omega^a{}_c\omega^b{}_d\left(A_\mu^{cd} \mp i\varepsilon^{cdf}A_\mu^{(0)f}\right) + \omega^a{}_c\partial_\mu\omega^b{}_c\right) =$$
$$= \frac{1}{2}\left(2\omega^a{}_c\omega^b{}_d A_\mu^{cd} + \omega^a{}_c\partial_\mu\omega^b{}_c\right) = \omega^a{}_c\omega^b{}_d A_\mu^{(\pm)cd} + \frac{1}{2}\omega^a{}_c\partial_\mu\omega^b{}_c. \tag{380}$$

That why

$$2A_\mu^{(\pm)ab} = \omega^a{}_c\omega^b{}_d\left(2A_\mu^{(\pm)cd}\right) + \omega^a{}_c\partial_\mu\omega^b{}_c \tag{381}$$

and, in particular

$$2A_i^{(\pm)ab} = \omega^a{}_c\omega^b{}_d\left(2A_i^{(\pm)cd}\right) + \omega^a{}_c\partial_i\omega^b{}_c. \tag{382}$$

It means that the quantity $(2A_i^{(\pm)ab})$ in view of (382) and (373) transforms in the same way as $A_i^{ab} = \overset{3}{A}{}_i^{ab}$. So the corresponding field strength

$$\overset{3}{F}{}_{ik}^{ab}\left(2A_l^{(\pm)dc}\right) = \partial_i(2A_k^{(\pm)ab}) - \partial_k(2A_i^{(\pm)ab}) + 2A_i^{(\pm)ac}2A_k^{(\pm)cb} - 2A_k^{(\pm)ac}2A_i^{(\pm)cb} \tag{383}$$

is a three-dimensional tensor like the initial three-dimensional field strength

$$\overset{3}{F}{}_{ik}^{ab}(A) = \partial_i A_k^{ab} - \partial_k A_i^{ab} + A_i^{ac}A_k^{cb} - A_k^{ac}A_i^{cb}. \tag{384}$$

In view of (361)

$$\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df}) = 2F^{(\pm)ab}_{ik}(A), \tag{385}$$

where $F^{(\pm)ab}_{ik}$ is the three-dimensional part of the (anti)selfdual field strength

$$F^{(\pm)AB}_{\mu\nu}(A) = \frac{1}{2}\left(F^{AB}_{\mu\nu}(A) \pm {}^*F^{AB}_{\mu\nu}(A)\right), \tag{386}$$

constructed of the initial connection A^{AB}_{μ}. Unlike the part $\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df})$ of the quantity (362), which was introduced in arbitrary $SO(1,3)$ frame and was only formally a three-dimensional field strength, constructed of $(2A^{(\pm)df})$, we have here $\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df})$, which is really three-dimensional field strength on $x^0 = const$ surface and is constructed of new three-dimensional connection $2A^{(\pm)df}_i$, which transforms as initial three-dimensional connection $A^{ab}_i = \overset{3}{A}{}^{ab}_i$.

So we get the following result. If one constructs on the surface $x^0 = const$ in a frame system with $e^{\mu}_{(0)} = n^{\mu}$ a doubled three-dimensional part $2A^{(\pm)ab}_i$ of the (anti)selfdual connection $A^{(\pm)AB}_{\mu} = \frac{1}{2}(A^{AB}_{\mu} \pm {}^*A^{AB}_{\mu})$, then this quantity $2A^{(\pm)ab}_i$ will be, in accordance with its transformation rule, a new three-dimensional connection, and the new three-dimensional field strength $\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df})$ coincides with $2F^{(\pm)ab}_{ik}(A)$, where $F^{(\pm)ab}_{ik}(A)$ is the set of all three-dimensional components of (anti)selfdual part of the field strength $F^{AB}_{\mu\nu}(A)$, based on initial connection A^{AB}_{μ}. Let us note that the three-dimensional strength

$$\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df}) = 2F^{(\pm)ab}_{ik}(A) = 2F^{ab}_{ik}(A^{(\pm)}) \tag{387}$$

does not coincide with the three-dimensional part of the four-dimensional quantity $F^{AB}_{\mu\nu}(2A^{(\pm)})$, which is not a Lorentz tensor in A, B indices, in spite of its selfduality. One need to avoid confusion between three-dimensional part of $F^{AB}_{\mu\nu}(2A^{(\pm)})$ and three-dimensional tensor $\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df})$.

One may represent the three-dimensional tensor $\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df})$ and the new three-dimensional connection $2A^{(\pm)df}_i$ as

$$\overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)df}) = \varepsilon^{abc}\overset{3}{F}{}^{c}_{ik}(2A^{(\pm)df}), \tag{388}$$

$$2A^{(\pm)ab}_i = \varepsilon^{abc}2A^{(\pm)c}_i, \tag{389}$$

where in accordance with (374)

$$2A^{(\pm)c}_i = \frac{1}{2}\varepsilon^{cab}2A^{(\pm)ab}_i = \varepsilon^{cab}\frac{1}{2}\left(A^{ab}_i \mp i\varepsilon^{abc}A^{(0)c}_i\right) = A^c_i \mp iA^{(0)c}. \tag{390}$$

Here we put as previously

$$A^c_i = \frac{1}{2}\varepsilon^{cab}A^{ab}_i. \tag{391}$$

So

$$2A^{(\pm)c}_i = A^c_i \mp iA^{(0)c}_i, \tag{392}$$

$$2A_i^{(\pm)ab} = \varepsilon^{abc} 2A_i^{(\pm)c} = \varepsilon^{abc}\left(A_i^c \mp iA_i^{(0)c}\right). \tag{393}$$

Let us compare this result with previously displayed canonical theory, in which B_i^c and Π_a^i are generalized coordinates and momenta. In view of (22)

$$A_i^{(0)c} = A_i^{(0)}{}_c = e_\alpha^{(0)}\Gamma^\alpha_{i\beta}e_c^\beta - (\partial_i e_\beta^{(0)})e_c^\beta. \tag{394}$$

In special tetrad system (37), (45), (48)

$$e_a^0 = 0, \quad e_i^{(0)} = 0, \quad e_{(0)}^0 = \frac{1}{N}, \quad e_0^{(0)} = N \tag{395}$$

and

$$A_i^{(0)c} = e_0^{(0)}\Gamma^0_{ik}e_c^k - (\partial_i e_k^{(0)})e_c^k = N\Gamma^0_{ik}e_c^k. \tag{396}$$

In view of (MT108) $N\Gamma^0_{ik} = -K_{ik}$, so

$$A_i^{(0)c} = -K_{ik}e_c^k. \tag{397}$$

In view of (MT348), (MT350)

$$K_{ik} = -\frac{1}{2}\mathcal{I}_{ik,lm}P^{lm}, \tag{398}$$

where P^{lm} are generalized ADM momenta. In view of (MT357)

$$\pi_{ik} = -\frac{1}{(2\varkappa)2\sqrt{\beta}}\mathcal{I}_{ik,lm}P^{lm}, \tag{399}$$

where π_{ik} are generalized FP momenta. By (398), (399)

$$\pi_{ik} = \frac{1}{(2\varkappa)\sqrt{\beta}}K_{ik}. \tag{400}$$

In view of (75) and (400)

$$\mathcal{P}_i^a = 2\pi_{ik}Q_a^k = \frac{2}{(2\varkappa)\sqrt{\beta}}K_{ik}Q_a^k = \frac{1}{\varkappa}K_{ik}e_a^k, \tag{401}$$

$$K_{ik}e_a^k = \varkappa\mathcal{P}_i^a. \tag{402}$$

In view of (392), (397) and (402)

$$2A_i^{(\pm)c} = A_i^c \pm i\varkappa\mathcal{P}_i^c. \tag{403}$$

If we compare (403) with the first equality (297) we see that they coincide, if we change up and down indices in (297) and set

$$B_i^c = 2A_i^{(\pm)c}. \tag{404}$$

It means that if $b = \pm i/\varkappa$, the dynamical variables B_i^c coincide with double three-dimensional part of (anti)selfdual component $A_\mu^{(\pm)AB} = \frac{1}{2}\left(A_\mu^{AB} + {}^*A_\mu^{AB}\right)$ of the connection A_μ^{AB} (upper sign in (403) for antiselfdual and down sign for selfdual components). Just that circumstance results in abrupt simplification of the constraint $\mathcal{H}_0$ under such values of b. It is clear that it is possible only in tetrad system with $e^{\mu}_{(0)} = n^{\mu}$.

Ashtekar got his construction beginning from selfdual tetrad connection and then he discovered essential simplification of $\mathcal{H}_0$ in this case. It became clear later, that equalities (158), (159) take place, and we lay this in the basis of our consideration. As we see, if $b = \pm\frac{i}{\varkappa}$ and $B^a_i = 2A^{(\pm)a}_i$, the relation

$$\overset{3}{F}{}^{ab}_{ik}(B) = \overset{3}{F}{}^{ab}_{ik}(2A^{(\pm)cd}) = 2F^{(\pm)ab}_{ik}(A) \qquad (405)$$

takes place, where $F^{(\pm)ab}_{ik}(A)$ is the three-dimensional part of the tensor

$$F^{(\pm)AB}_{\mu\nu}(A) = \frac{1}{2}\left(F^{AB}_{\mu\nu} \pm {}^*F^{AB}_{\mu\nu}\right), \qquad (406)$$

and $\overset{3}{F}{}^{ab}_{ik}(B)$ is specified by (245).

As we noted before, complex tetrad (anti)selfdual formalism is in use more rarely, than real non(anti)selfdual theory with more involved constraint $\mathcal{H}'_0$ (or $\mathcal{H}''_0$) because of very complicated return from complex to real region.

Hereon we complete the description of the different forms of canonical tetrad formalism in gravitation theory.

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